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Applications of Wave-Optical Weak Lensing of Gravitational Wave

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Based on

- "Small-scale shear: peeling off diffuse subhalos with gravitational waves"
  Han Gil Choi, Chanung Park and Sunghoon Jung, Phys. Rev. D **104**, 063001 (2021)

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2. Probing $\boldsymbol{P_m(k)}$ through combining weak lensing events

Probing small dark matter halos with weak diffractive lensing

Based on "Small-scale shear: peeling off diffuse subhalos with gravitational waves"

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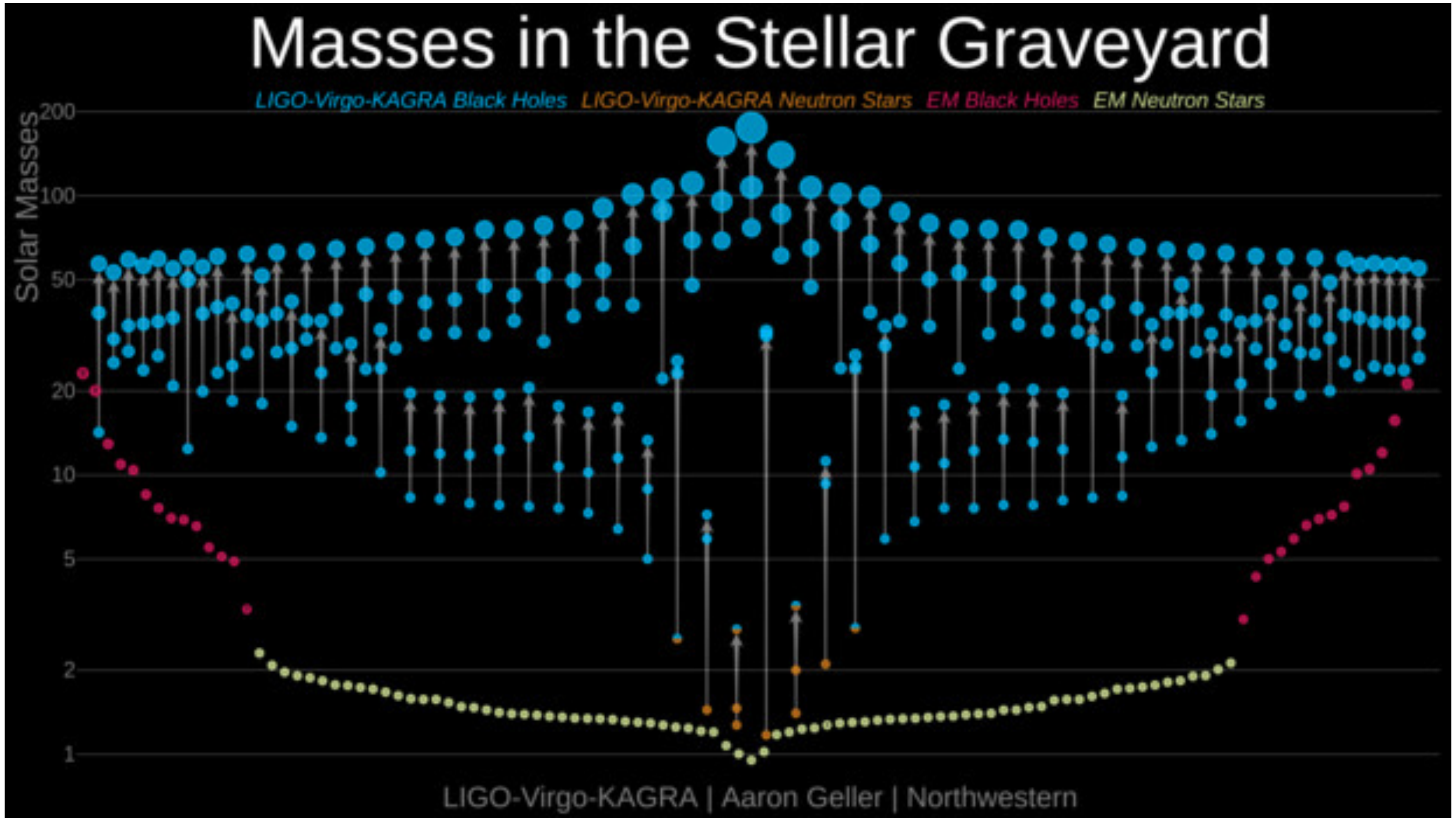
Motivations – GW

GW can be used to probe dark universe
By Gravitational lensing of GW(GW lensing)



Motivations - GL

- Gravitational Lensing
  - **Strong lensing**
  - Weak lensing

Credit :NASA,ESA

Motivations - GL

- Gravitational Lensing
  - Strong lensing
  - **Weak lensing**

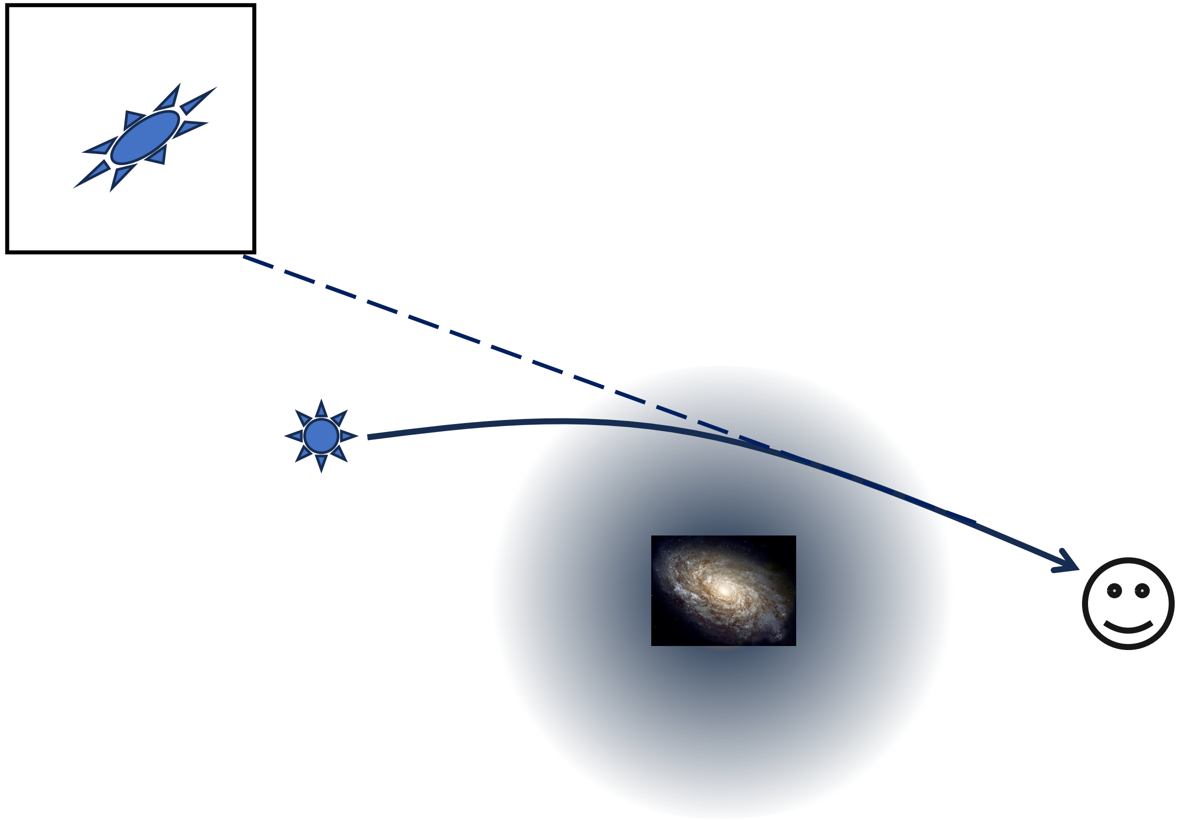
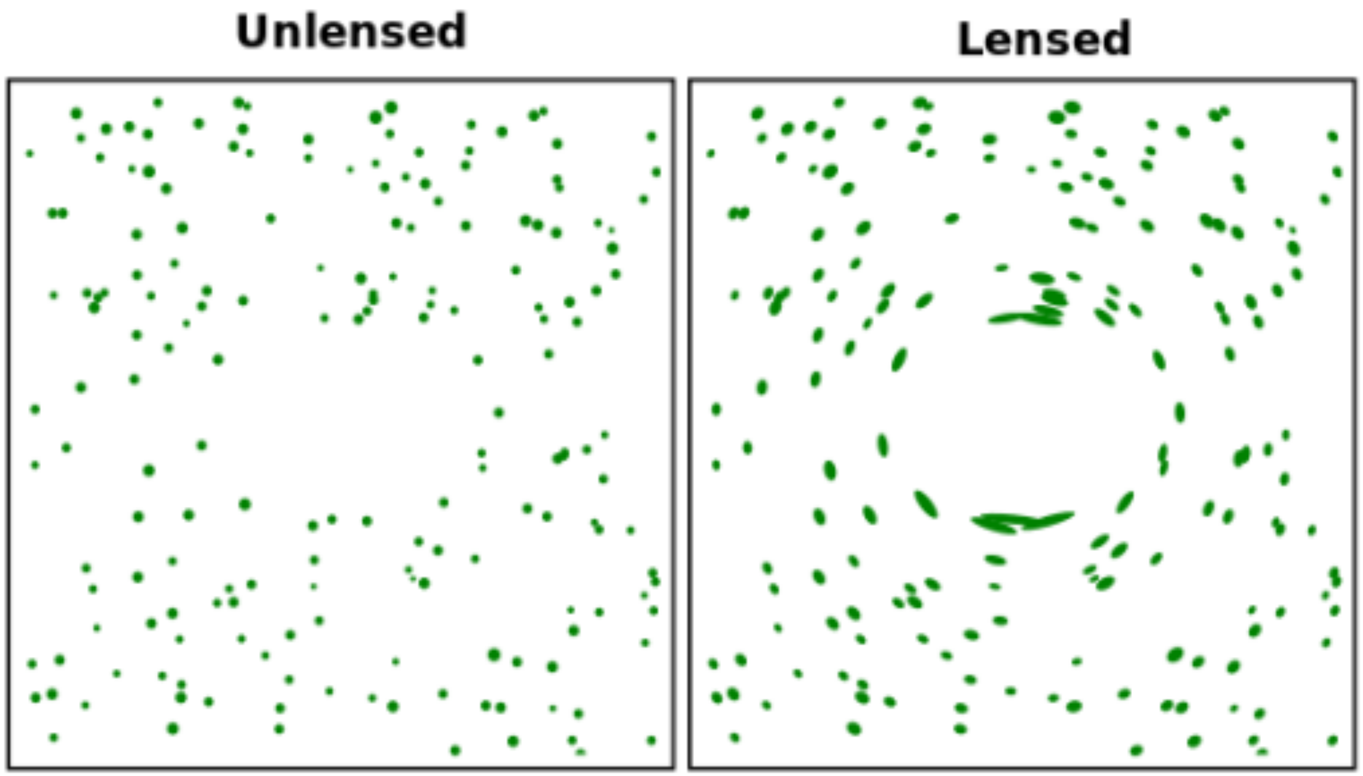


Image deformation
$\propto$ Line-of-Sight mass density(Convergence) & Tidal effect(Shear)

However, Single source does not give any information – Unknown intrinsic properties

Motivations - GL

Many Sources → Field of convergence & shear → Lens profile!



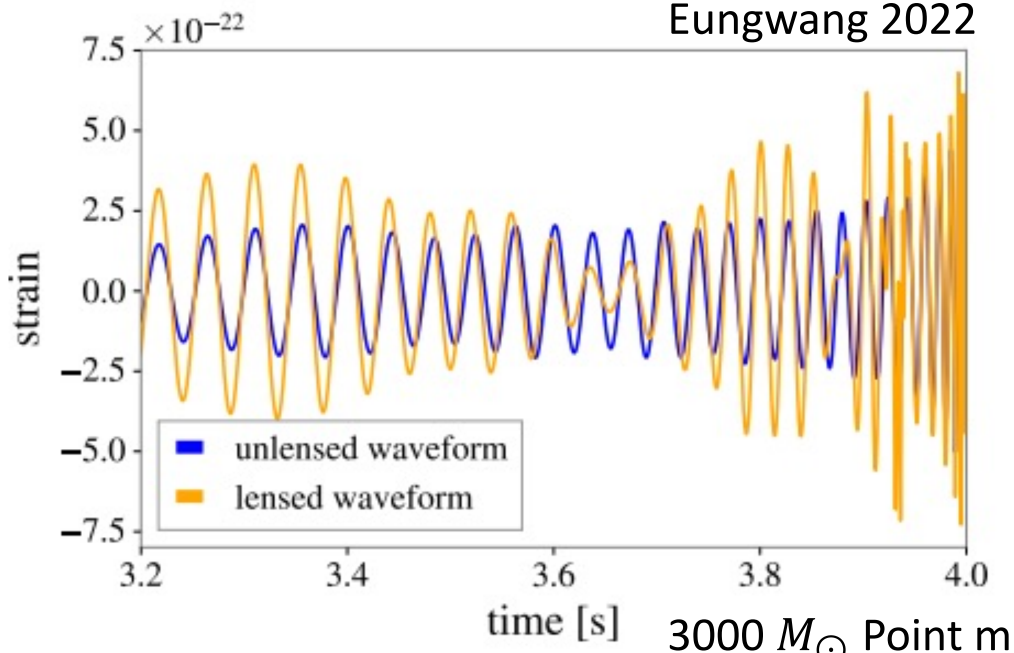
Motivations - GL

- Strong GW lensing has been used to probe compact lens objects (ex. black holes)

$$h_L(t) = h_1(t) + h_2(t) + \cdots$$

Similar GWs with different arrival timing and phase

Eungwang 2022

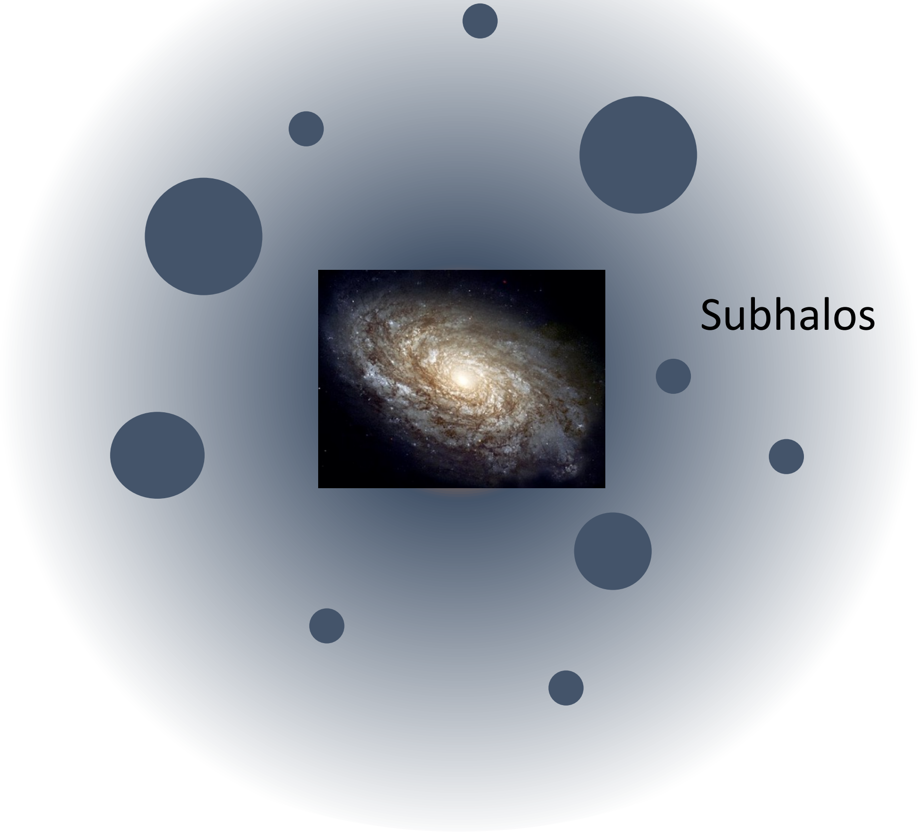


$3000\ M_{\odot}$ Point mass lens

Distortion of signal by interference between multiple images

Motivations – Dark matter halo

- Can we detect more diffuse lens object like **Dark matter halo**?
- **Small Dark matter halo can give a hint on dark matter properties.**
- Indirect evidence of small dark matter (sub)halo
  - $\mathrm{M_{sub}} > 10^{7} M_{\odot}$ (Nadler 2021)
- We want to lower the limit by GW lensing.



Motivations – Wave optics

- However, small halos below M $\leq 10^6 M_\odot$ are **too diffuse**, they produce only **weak lensing.** $h_L(t) = h_1(t)$

- Effects of weak lensing in **Geometric optics** calculation

$$h_L(t) = \sqrt{\mu} h(t - \tau)$$

- Amplification : **unobservable** unless we know the exact distance to the source
- Time-delay : **unobservable** unless we know the exact GW emission timing

- Only **Diffraction** gives observables.
  - We need to solve the wave equation of GW.

Wave optics of GW

- Background metric in weak gravity

$$ds^2 = -(1 + 2U(\mathbf{x}))dt^2 + (1 - 2U(\mathbf{x}))d\mathbf{x}^2 = g^{(B)}_{\mu\nu} dx^\mu dx^\nu$$

$$\nabla^2 U = 4\pi\rho$$

- Propagation of GW (with appropriate gauge fixing)

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu} \qquad \Box^{(B)} h_{\mu\nu} + 2R^{(B)}_{\gamma\mu\delta\nu} h^{\gamma\delta} = 0$$

- Wavelength << Background Curvature length scale

$$\Box^{(B)} h_{\mu\nu} \simeq 0$$

- Negligible changes(~U) in the polarizations

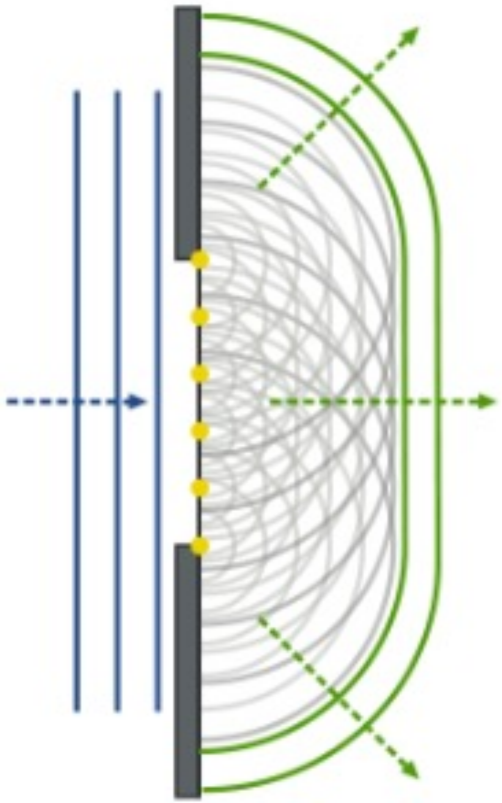
$$h_{\mu\nu}(t, \mathbf{x}) \simeq \phi(t, \mathbf{x}) e_{\mu\nu}$$

$$(\nabla^2 + w^2)\phi(w, \mathbf{x}) = 4w^2 U(\mathbf{x})\phi(w, \mathbf{x})$$ * Fourier transform

Wave optics of GW

- The solution is given by Kirchhoff's diffraction formula
  - Huygens' principle on lens plane

cf) Diffraction by a single Slit

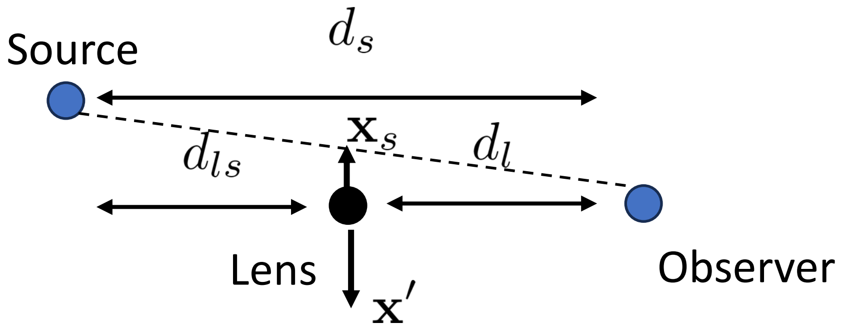


Wave optics of GW

- The solution is given by Kirchhoff's diffraction formula
  - Huygens' principle on lens plane

$$\frac{\phi(w;\mathbf{x}_o)}{\phi_0(w;\mathbf{x}_o)} = F(w;\mathbf{x}_s) = \frac{1}{2\pi i}\int \frac{dx'^2}{r_F^2(w)} e^{i\left[\frac{1}{2}|\mathbf{x}'-\mathbf{x}_s|^2-\psi(\mathbf{x}')\right]/r_F^2(w)}$$

$$r_F = \sqrt{\frac{d_{\rm eff}}{w}}$$ **Fresnel length**



$$d_{\rm eff} = \frac{d_l d_{ls}}{d_s}$$ Effective lens distance

$$\psi(\mathbf{x}) = 2d_{\rm eff}\int dz U(\mathbf{x},z)$$ Line-of-sight gravitational potential

Weak Diffractive lensing

Wave optics in Diffraction regime: $\mathbf{x}_s \ll r_F$

$$\Rightarrow F(w;\mathbf{x}_s) \propto \int dx'^2 \exp\left[i\left[\frac{1}{2}|\mathbf{x}|^2 - \psi(\mathbf{x})\right]\frac{1}{r_F^2}\right]$$

Weak lensing approximation: **linear in $\boldsymbol{\psi}$** & low slope profile

$$F(w) \simeq 1 + \overline{\kappa}(e^{i\frac{\pi}{4}} r_F)$$ Aperture mean convergence

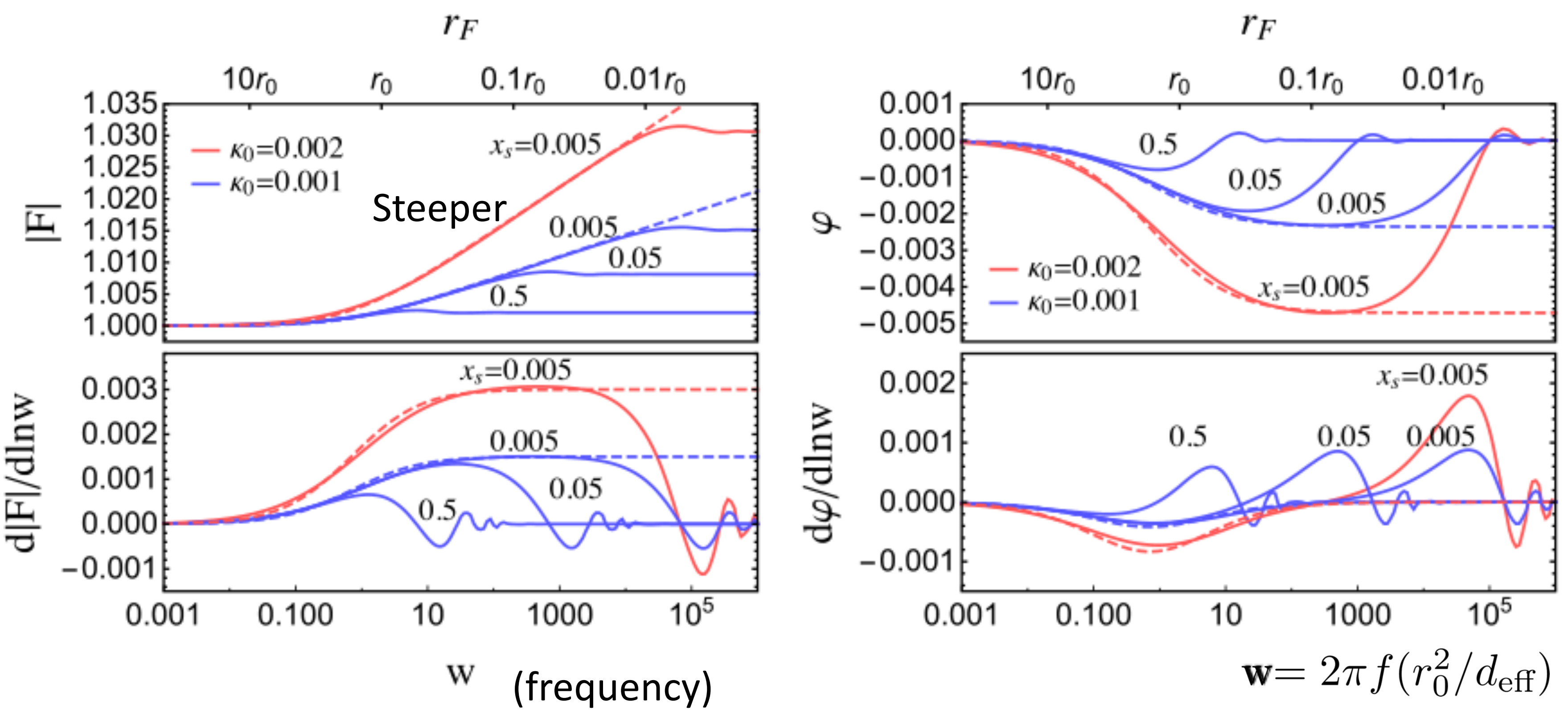
Frequency dependency is observable of diffraction:

$$\frac{dF(w)}{d\ln w} \simeq \gamma_t(e^{i\frac{\pi}{4}} r_F)$$ (Tangential) Shear

F(w) is directly related to lensing profile at Fresnel length!

Weak Diffractive lensing

- F(w) of Navarro-Frenk-White(NFW) profile. Numerical vs Analytic
- Good matches when $r_F > x_s$
- The **slope of F(w) follows the slope** of the DM halo profile.



Detection of Diffractive lensing

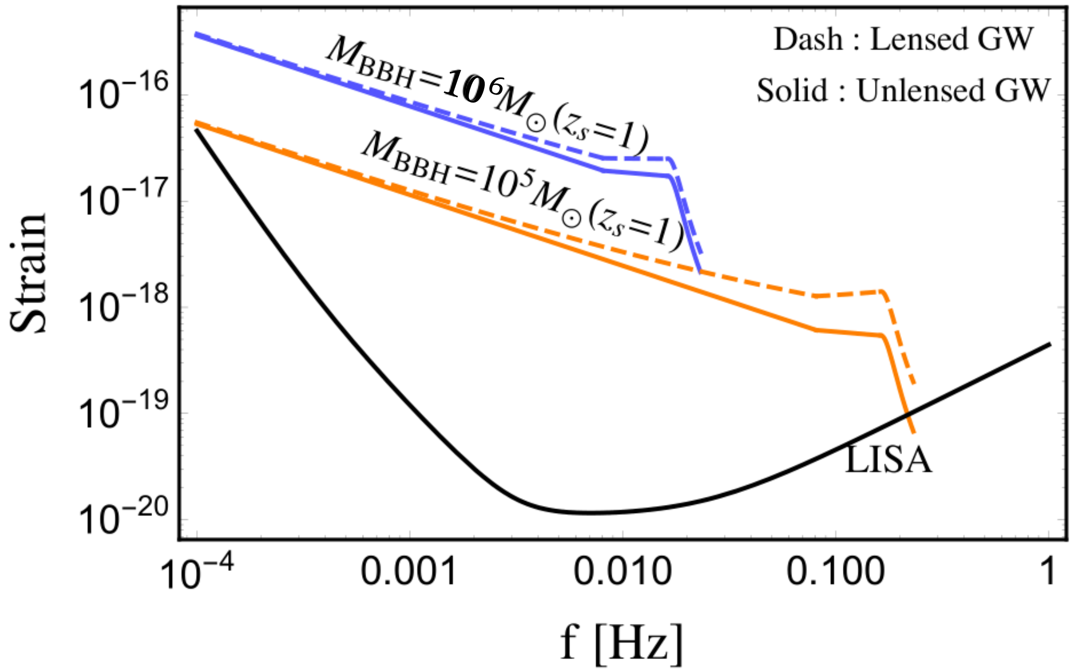
Multiple frequencies replace multiple background sources

Gaussian Noise, Small detector noise, Ignore correlations with source intrinsic parameters

$$\text{log-likelihood} \simeq \min_{A,\phi,t} 2 \sum_{f_j} \frac{|F(f_j) - Ae^{i\phi}e^{2\pi i f_j t}|^2 |h_0(f_j)|^2}{S_n(f_j)} \Delta f$$

Lensing by Singular Isothermal sphere lens ($M = 10^5\, M_\odot, z_l = 0.35$)

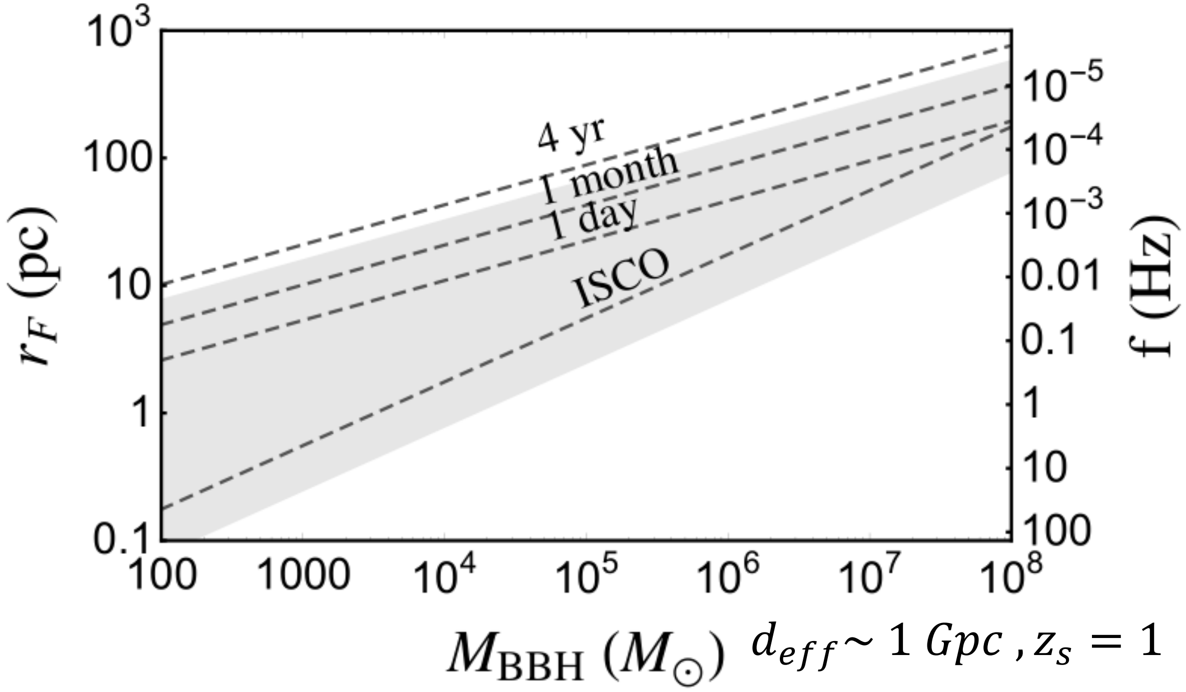
Ex) spectrum change



Detection of Diffractive lensing

- Small DM halo with **10 pc** length scale can be probed by **Massive BBH mergers**

$$r_F \equiv \sqrt{\lambda d_l} \simeq 5.56\text{pc}\sqrt{\left(\frac{d_l}{\text{Gpc}}\right)\left(\frac{0.1\text{Hz}}{f}\right)}$$

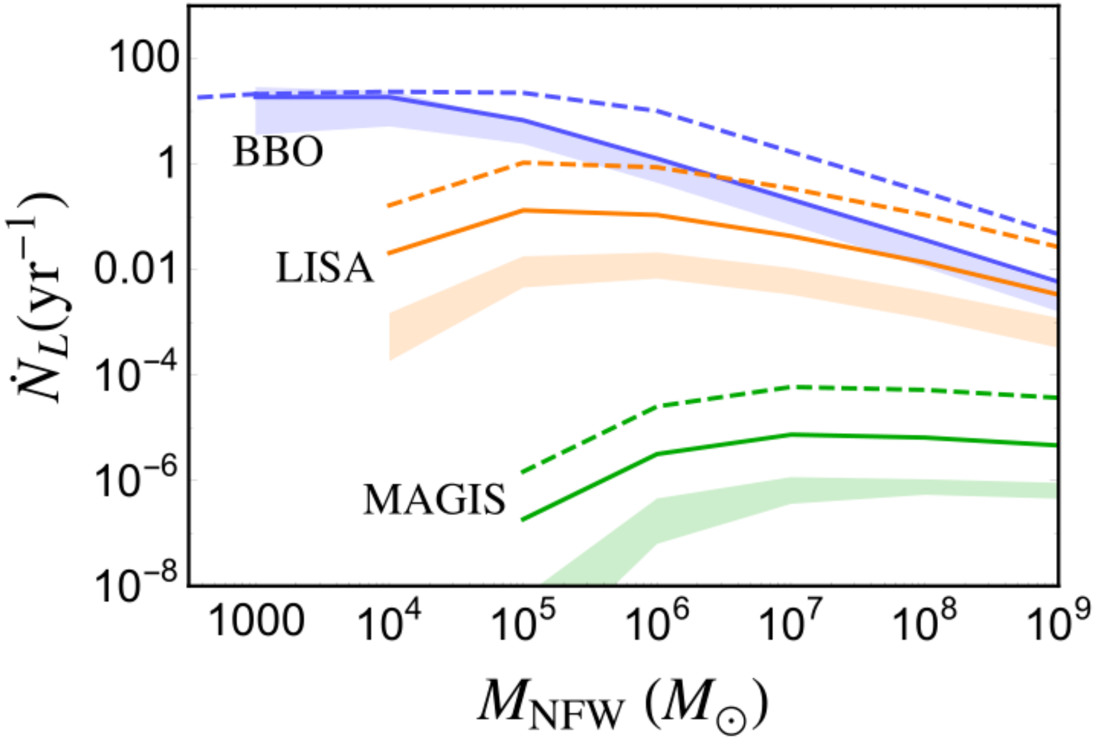


$d_{eff} \sim 1\ Gpc$, $z_s = 1$

Prospects

- Big Bang Observer (BBO) can detect **(CDM)** $\mathbf{10^3 M_\odot}$ **halo** more than 10 per year.
  - The others are less promising due to strong detector noise.

- The prospects highly depends on **massive BBH merger population** and **DM halo population**.



BBH Merger rate
Solid : 0.01 $Gpc^{-3}yr^{-1}$
Shaded : astrophysical (Bonetti 2018)

Dashed : No halo mass distribution

Probing $\boldsymbol{P_m(k)}$ through combining weak lensing events

Work in progress

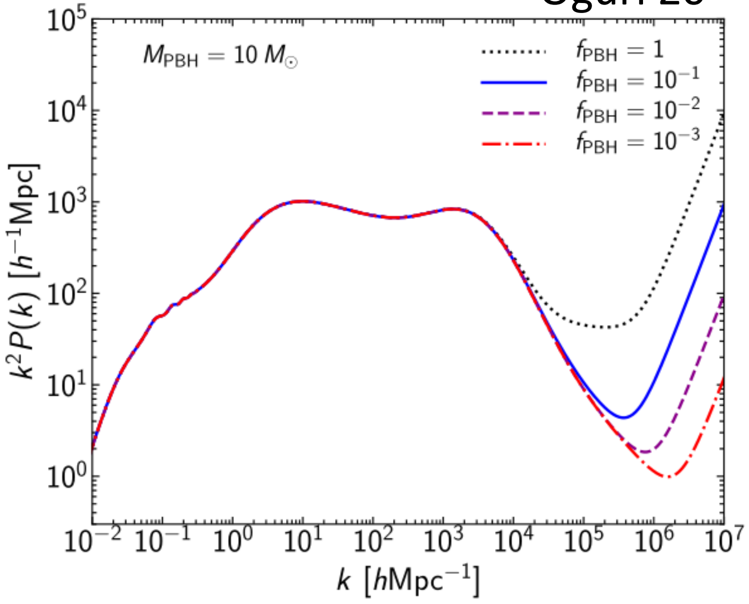
Lensing signal from $\boldsymbol{P_m(k)}$ Work in progress

- Some dark matter models predict **shot noise dominance at small scale(< 1 pc)**
  - **Primordial black hole**, Axion mini halo …

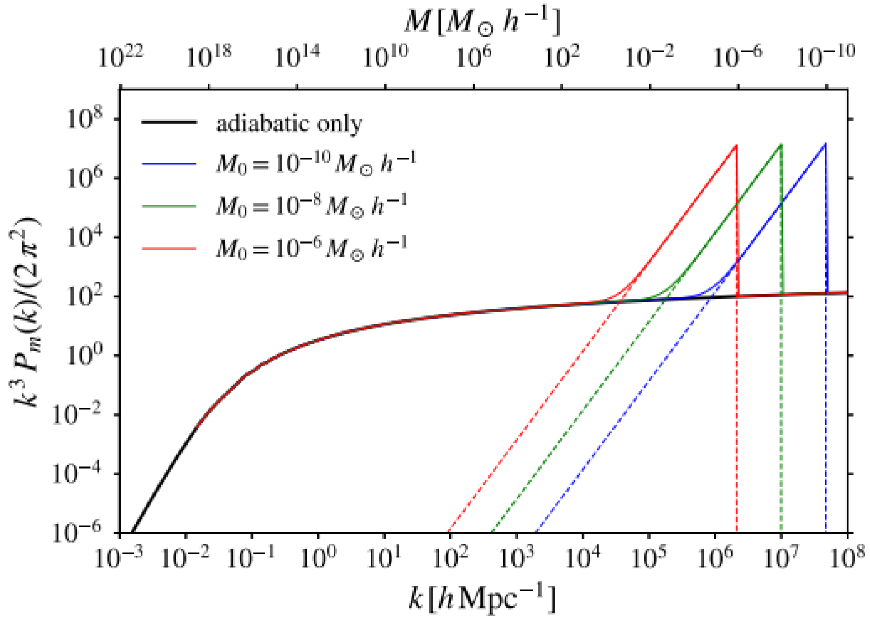
$$P_{\rm shot}(k) = \frac{f^2}{\overline{n}}$$

$\overline{n}$: number density
f : mass fraction

PBH Oguri 20



Axion mini-halo Dai 20



Lensing signal from $\boldsymbol{P_m(k)}$ Work in progress

- In the high frequency detectors, GW lensing events are rare(probability $\leq$ 0.001).

- Lens model dependent (point mass, halo, etc.)

- Instead, How about combine **all GW events**?
  - Lens effects (= gravitational potentials) become stochastic.
  - The stochastic nature is related to **matter power spectrum.**
  - LIGO (ET) will observe >100 (10^5) GW events per year.
  - cf. Stochastic Gravitational Wave Background
    - accumulation of stochastic detector correlations

- LIGO and ET are sensitive to BBH merger at 100 Hz
  - Ultra small scale!

$$r_F \propto \sqrt{\lambda d_l} \simeq 0.18\text{pc}\sqrt{\left(\frac{d_l}{\text{Gpc}}\right)\left(\frac{100\text{Hz}}{f}\right)}$$

Lensing signal from $P_m(k)$ Work in progress

- Strain data consist of detector noise, GW signal, and lensing signal

$$d(f) = n(f) + h_0(f) + \delta F \cdot h_0(f)$$

- Lensing effects of all Line-of-Sight matter inhomogeneity

$$\delta F(f) = \frac{4\pi f}{i} \int_0^{d_s} dx \int \frac{d^3k}{(2\pi)^3} \tilde{U}(\mathbf{k}) e^{-ik_{||}(d_s - x)} e^{-i\frac{k^2 x(d_s - x)}{4\pi f d_s}}$$

- The lensing amplification factor has zero mean. $\langle \delta F \rangle = 0$

- **The covariance of the residuals** is related to **the matter power spectrum**

$$\Big\langle (d^*(f_1) - h_0^*(f_1))(d(f_2) - h_0(f_2)) \Big\rangle = \langle (\delta F \cdot h_0(f_1))^* \delta F \cdot h_0(f_2) \rangle \propto P_m h_0^2$$

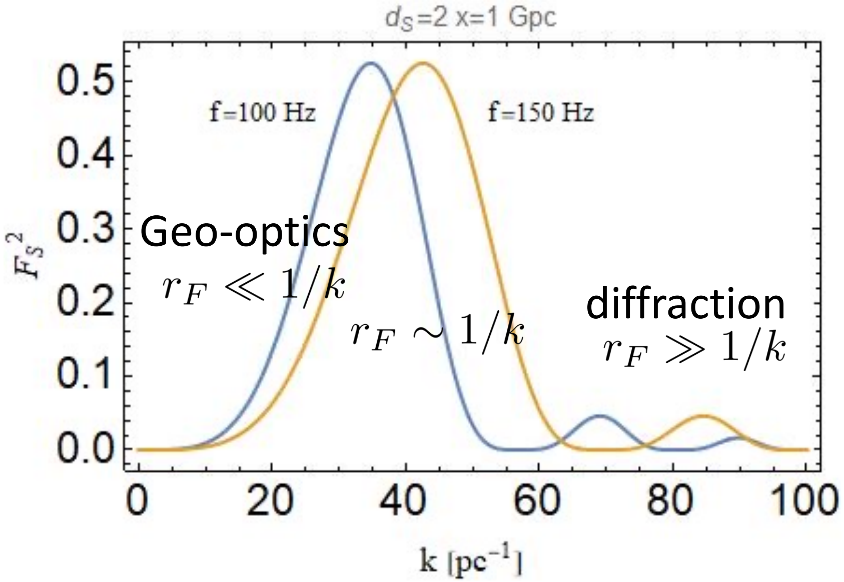
*The stationary detector noise is assumed

Lensing signal from $P_m(k)$ Work in progress

ex) $$\text{GW phase fluctuation} = \int_0^{d_s} dx W^2(x) \int \frac{dkk}{2\pi} P_m(k) F_S^2(k,x)$$

$$W(x) \equiv x\left(1 - \frac{x}{d_s}\right) \qquad F_S(k,x) = \frac{\cos(kr_F)^2/2 - 1}{(kr_F)^2/2}$$

- The fluctuations exists only when $r_F \sim 1/k$.
- There is **no fluctuation** for geo-optics limit ($r_F \ll 1/k$) and diffraction limit ($r_F \gg 1/k$). (Oguri 20)



Detecting lensing signal

- We estimate the Bayes factor
  - Lensing vs no lensing Hypothesis

$$B \equiv \frac{\mathcal{L}_{\rm lensing}}{\mathcal{L}_{\rm no\ lensing}} \qquad \mathcal{L} \propto e^{-2\int df |d(f)-h(f)|^2/S_n(f)}$$

$$B \equiv \frac{\int d\theta \prod_i dU_i p(d|h_0(\theta), \{U_i\}, H_1)\pi(\theta, \{U_i\}|H_1)}{\int d\theta p(d|h_0(\theta), H_0)\pi(\theta|H_0)}$$

- Expectation value of the **lnB is zero** for Geo. limit and diffraction limit

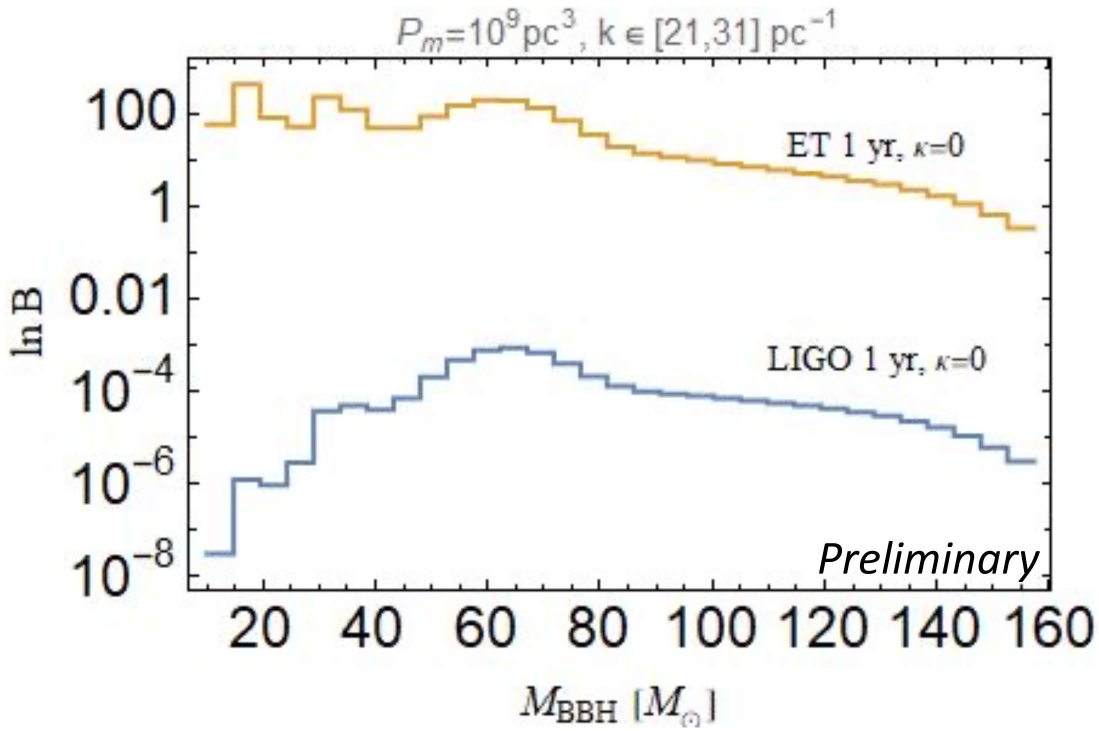
$$\langle \ln B \rangle \propto \int_0^{d_s} dx_1 W^2(x_1) \int \frac{dk_1 k_1}{2\pi} P_m(x_1, k_1) \times \int_0^{d_s} dx_2 W^2(x_2) \int \frac{dk_2 k_2}{2\pi} P_m(x_2, k_2) \times \rho_0^4 \left[{\rm cov}(F_{S1}, F_{S2}) - \sigma_w^{-2} {\rm cov}(w, F_{S1}) {\rm cov}(w, F_{S2}) \right]^2$$

After marginalized over amplitude, phase, time shift freedom…

$$ {\rm cov}(A, B) = \overline{AB} - \overline{A}\,\overline{B} \qquad F_{S1,2} \equiv F_S(k_{1,2}, x_{1,2})$$

Detecting lensing signal

- We can combine all GW data : $\ln B = \sum_i \ln B_i$
- We consider stellar mass BBH merger events
  - * we assume the *merger rate* distribution from LIGO-VIRGO O3 data
- In LIGO, the contribution mainly comes from $M_{BBH} \sim 60 M_\odot$ while in ET, $M_{BBH} \sim 10 - 60 M_\odot$

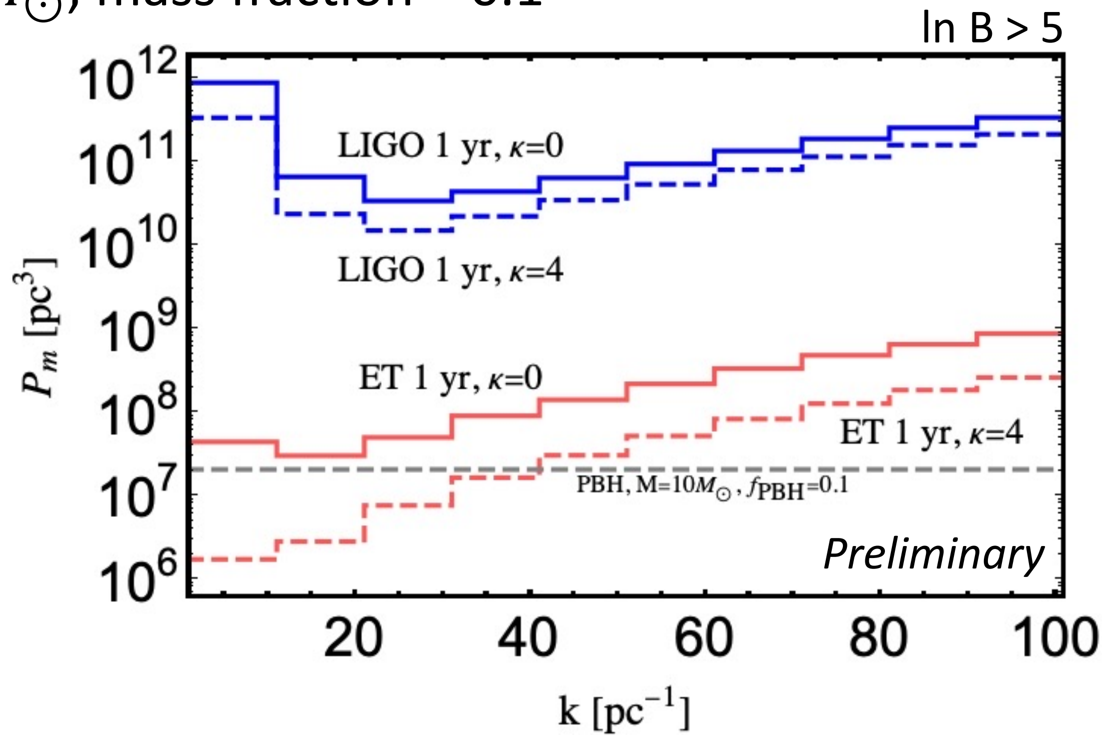


Prospects

- We assume $P_m(k,z) = P_m^0(k)(1+z)^{\kappa-3}$
  - P_m is constant within the bins

- Best constraint is at
  - LIGO : $k = 30\, pc^{-1}$, ET : $k = 10\, pc^{-1}$

- 1yr observation with ET can **reach the PBH shot noise** level !
  - PBH mass =10 $M_\odot$, mass fraction = 0.1



Constraining power

Summary

1. Diffractive lensing is controlled by the Fresnel length r_F which is frequency and distance dependent.

2. r_F of GW from massive BBHs can be few parsecs. Therefore, light sub halos can be detected through diffractive lensing.

3. Powerful mid-band GW detector like BBO can detect few tens of $10^{3\sim4}M_{\odot}$ DM halo per year.

4. Combining many GW data of LIGO and ET, the matter power spectrum at sub-parsec scale can be probed.