Curing ghosts with inequality constraints

Pavel Jiroušek

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UNIVERSITY OF CAPE TOWN
IYUNIVESITHI YASEKAPA • UNIVERSITEIT VAN KAAPSTAD

Work in progress

Work in progress
with Alex Vikman

coming to arXiv (hopefully) in 2023

Spoilers

- There is significant interest in higher derivative theories in modified gravity
- Extra derivatives $\rightarrow$ extra degrees of freedom
- Hamiltonian becomes unbounded from bellow

$$H \ngeq E_0$$

- Healthy systems can siphon arbitrary amount of energy from this system $\rightarrow$ instability?
- Inequality constraints A. Frolov, V. Frolov, 2304.12179

$$\Phi = 0 \rightarrow \Phi \geq 0$$

- Can we use this to fix

$$H \geq E_0$$

- How to do this?
- What does it mean for the system?

Inequality constraint

- Normal constraint

$$\Phi(q, \dot{q}, \dots) = 0$$

- Inequality constraint

$$\Phi(q, \dot{q}, \dots) \geq 0$$

- Such a constraint can be imposed by the following ordinary Lagrange constraint

$$L = L_0(q, \dot{q}, \dots) + \lambda \left(\Phi(q, \dot{q}, \dots) - \xi^2 \right)$$

Inequality constraints

- Equations of motion for q

$$\frac{\delta L_0}{\delta q} + \lambda \frac{\delta \Phi}{\delta q} = 0$$

- The variation with respect to λ

$$\Phi(q, \dot{q}, \dots) = \xi^2$$

- The variation with respect to ξ

$$\lambda \xi = 0$$

this has two branches of solution

$$\lambda = 0\ , \qquad \text{and} \qquad \xi = 0$$

Subcritical phase

- The subcritical phase

$$\lambda = 0$$

- The constraint determines ξ^2

$$\Phi(q, \dot{q}, \dots) = \xi^2$$

this is only possible when

$$\Phi(q, \dot{q}, \dots) \geq 0$$

- For $\lambda = 0$ there is no modification of the original evolution

$$\frac{\delta L_0}{\delta q} = 0$$

Supercritical phase

- Supercritical phase

$$\xi = 0$$

- The constraint is saturated

$$\Phi(q, \dot{q}, \dots) = 0$$

- The system behaves as if the constraint $\Phi = 0$ is enforced directly

$$\frac{\delta L_0}{\delta q} + \lambda \frac{\delta \Phi}{\delta q} = 0$$

Transition

- The original evolution may take us from

$$\Phi > 0 \ \text{ to } \ \Phi < 0$$

  the second part is now prohibited, but we have a possibility to 'escape' to the supercritical phase
- Is this transition smooth? - No
- In the subcritical phase $\dot{\Phi} = 2\xi\dot{\xi} \neq 0$ and $\dot{\lambda} = 0$
- In the supercritical phase $\dot{\xi} = 0$ and $\dot{\lambda} \neq 0$
- Generally the transition is discontinuous in these velocities!

Compatibility of the constraint

- Continuity of the transition
- For example take the system

$$L = \frac{\dot{q}^2}{2} - \frac{q^2}{2} + \lambda\left(q - q_c - \xi^2\right)$$

- The subcritical phase of the evolution is just harmonic oscillator for $q(t) > q_c$

$$q(t) = q_0 \cos(t - t_0)$$

  along with

$$\lambda = 0\ , \qquad \xi^2 = q - q_c\ , \qquad p = -q_0 \sin(t - t_0)$$

- The supercritical phase gives

$$q = q_c\ , \quad \xi = 0\ , \quad \lambda = q_c\ , \quad p = 0$$

- The transition is discontinuous in the phase space
- The reason for this is because the two branches have different number of degrees of freedom! PJ, Shimada, Yamaguchi, Vikman, 2208.05951
- This will haunt us later on

Ostrogradsky ghost

- If we have a higher derivative Lagrangian

$$L = L(\ddot{q}, \dot{q}, q)$$

then we can introduce the following generalization of the Legandre map

$$P = \frac{\partial L}{\partial \ddot{q}}(\ddot{q}, \dot{q}, q) \,, \qquad Q = \dot{q} \,,$$
$$p = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}} \,, \qquad q = q$$

- The Ostrogradsky Hamiltonian is then

$$\begin{aligned} H &= p\dot{q} + P\dot{Q} - L(\ddot{q}, \dot{q}, q) \\ &= pQ + H_0(P, Q, q) \end{aligned}$$

- The first term is necessarily unbounded in the momentum p
- Possible interactions are assumed to be part of H_0

Curing the ghost

- Let us apply the inequality constraint on the problematic term

$$H \to pQ + H_0(P,Q,q) + \lambda\left(pQ - E_0 - \xi^2\right)$$

This should implement the inequality

$$pQ \geq E_0$$

- To enforce the constraints, we impose two primary constraints

$$\pi_\lambda = 0$$
$$\pi_\xi = 0$$

and include them in the Hamiltonian

$$H \to pQ + H_0(P,Q,q) + \lambda\left(pQ - E_0 - \xi^2\right) + \alpha\pi_\lambda + \beta\pi_\xi$$

Curing the ghost - action

- The first order action

$$S = \int dt \left[p\dot{q} + P\dot{Q} - pQ - H_0(P,Q,q) - \lambda \left(pQ - E_0 - \xi^2 \right) \right]$$

  which is varied with respect to p, q, P, Q, λ, ξ
- Note that p is a Lagrange multiplier enforcing

$$Q = \frac{\dot{q}}{1+\lambda}$$

- Integrate out p, P, Q, α, β

$$S = \int dt \left[L\left(\frac{d}{dt}\frac{\dot{q}}{1+\lambda}, \frac{\dot{q}}{1+\lambda}, q \right) + \lambda(\xi^2 + E_0) \right]$$

Curing the ghost - action

- Lorentz non-covariant modification

$$L(\ddot{\phi}, \dot{\phi}, \partial_i^2\phi, \partial_i\phi, \phi) \to L\left(\frac{d}{dt}\frac{\dot{\phi}}{1+\lambda}, \frac{\dot{\phi}}{1+\lambda}, \partial_i^2\phi, \partial_i\phi, \phi\right) + \lambda\left(\xi^2 + E_0\right)$$

- The subcritical phase is completely Lorentz covariant if the original was
- Symmetry breaking in the supercritical phase

Curing the ghost - dynamics

- Consistency of the primary constraints

$$\dot{\pi}_\lambda = \{\pi_\lambda, H\} = \xi^2 + E_0 - pQ = 0$$
$$\dot{\pi}_\xi = \{\pi_\xi, H\} = -2\lambda\xi = 0$$

- The second condition has two branches

$$\lambda = 0 \;, \qquad \text{subcritical phase}$$
$$\xi = 0 \;, \qquad \text{supercritical phase}$$

- Subcritical phase consistency

$$\dot{\lambda} = \{\lambda, H\} = \alpha = 0$$
$$\frac{d}{dt}\left(\xi^2 + E_0 - pQ\right) = \left\{\xi^2 + E_0 - pQ, H\right\} = p\frac{\partial H_0}{\partial P} + 2\beta\xi = 0$$

there are no further constraints

- There are two degrees of freedom

Curing the ghost - dynamics

- Supercritical phase consistency

$$\frac{d}{dt}\left(pQ - E_0\right) = \{pQ - E_0, H\} = \frac{\partial H_0}{\partial P}p - \frac{\partial H_0}{\partial q}Q = 0$$

$$\dot{\xi} = \{\xi, H\} = \beta = 0$$

$\rightarrow$New tertiary constraint

- Consistency

$$\left\{\frac{\partial H_0}{\partial P}p - \frac{\partial H_0}{\partial q}Q, H\right\} = (1+\lambda)\left\{\left\{pQ, H_0\right\}, pQ\right\} + \left\{\frac{\partial H_0}{\partial P}p - \frac{\partial H_0}{\partial q}Q, H_0\right\} = 0$$

- If $\left\{\left\{pQ, H_0\right\}, pQ\right\} \neq 0$ we can solve for λ

$$\lambda = \dots$$

we do not get any additional constraints

- There is now one degree of freedom!

Curing the ghost - transition

- As in the previous case $\rightarrow$ discontinuity in the transition
- The subcritical branch has two degrees of freedom and therefore 4 initial conditions
- The evolution in the subcritical phase 'lands' on the constraints surface $pQ = E_0$

  $\rightarrow$ 3 free parameters
- The evolution on this constraint can only be consistent when

$$\frac{\partial H_0}{\partial P} p - \frac{\partial H_0}{\partial q} Q = 0$$

- Forcing the transition we have to project to this constraint.
- $(q, Q) \sim (q, \dot{q})$ continuous
- $(P, p) \sim (\ddot{q}, \dddot{q})$ discontinuous

Fixing the transition

- We can fix this by modifying the procedure

$$H \to H + \lambda \left(pQ + \{\Psi, H\} - E_0 - \xi^2\right) + \alpha(\pi_\lambda - \Psi) + \beta\pi_\xi$$

- Primary constraint

$$\begin{aligned} \pi_\lambda &= \Psi(p, q, P, Q) \\ \pi_\xi &= 0 \end{aligned}$$

- The consistency of primary constraint

$$\begin{aligned} \{\pi_\lambda - \Psi, H\} &= \xi^2 + E_0 - pQ - \lambda \{pQ - \{\Psi, H\}, \Psi\} + 0 \cdot \{\Psi, H\} = 0 \\ \{\pi_\xi, H\} &= -2\lambda\xi = 0 \end{aligned}$$

- The second condition has two branches

$$\lambda = 0 \; , \qquad \xi = 0$$

Fixing the transition

- Subcritical phase $\lambda = 0$

$$pQ - E_0 = \xi^2 \implies pQ - E_0 \geq 0$$

- Supercritical branch $\xi = 0$

$$\lambda = \frac{pQ - E_0}{\left\{pQ - \left\{\Psi, H\right\}, \Psi\right\}}$$

- The consistency in this phase now gives

$$\left\{E_0 - pQ + \lambda \left\{\left\{\Psi, H\right\}, \Psi\right\}, H\right\} = \alpha \cdots = 0$$

No additional constraints

- There are four degrees of freedom

Fixing the transition

- The Hamiltonian in the supercritical phase

$$H|_{sup} = H_0 + pQ\left(1 + \frac{pQ - E_0}{\{pQ - \{\Psi, H\}, \Psi\}}\right)$$

- Is this bounded from bellow?
- This gives a condition on Ψ

$$pQ\left(1 + \frac{pQ - E_0}{\{pQ - \{\Psi, H\}, \Psi\}}\right) \geq \mathcal{E}_0$$

- A possible way to achieve this is to have

$$\{pQ - \{\Psi, H\}, \Psi\} = 1$$

Example

- Suppose
$$H_0 = \bar{H}_0(P, Q) + \frac{q^2}{2}$$
and
$$\Psi = p$$
- With these choices we indeed have
$$\{pQ - \{\Psi, H\}, \Psi\} = -\left\{\left\{p, \frac{q^2}{2}\right\}, p\right\} = 1$$
- The Hamiltonian is then
$$H = \bar{H}_0 + \frac{q^2}{2} + pQ + \lambda\left(pQ + q - E_0 - \xi^2\right) + \alpha(\pi_\lambda - p) + \beta\pi_\xi$$
- In the supercritical phase
$$H|_{sup} = \bar{H}_0 + \frac{q^2}{2} + pQ + (pQ - E_0)(pQ + q - E_0)$$

Example - Lagrangian

- The above Hamiltonian in general arises from the Lagrangian

$$L = \bar{L}(\ddot{q}, \dot{q}) - \frac{q^2}{2}$$

- The first order action

$$S = \int dt \left[p(\dot{q} + \dot{\lambda}) + P\dot{Q} - pQ - \bar{H}_0(P, Q) - \frac{q^2}{2} - \lambda \left(pQ + q - E_0 - \xi^2 \right) \right]$$

- The momentum p is a Lagrange multiplier enforcing

$$Q = \frac{\dot{q} + \dot{\lambda}}{1 + \lambda}$$

Hence the Lagrangian is

$$L = \bar{L} \left(\frac{d}{dt} \frac{\dot{q} + \dot{\lambda}}{1 + \lambda}, \frac{\dot{q} + \dot{\lambda}}{1 + \lambda} \right) - \frac{q^2}{2} + \lambda \left(E_0 + \xi^2 - q \right)$$

Conclusions

- The idea is to bound unbounded (ghosty) Hamiltonians using the inequality constraints
- This can be done using two auxiliary variables (fields) λ and ξ
- The modified system has two phases
  - Subcritical phase - no change to the original behavior (up to initial conditions)
  - Supercritical phase - changes to the dynamics, Lorentz symmetry breaking
- The transition between the phases is almost guaranteed to be discontinuous in velocities
- The amount of discontinuities depends on the amount of degrees of freedom in the supercritical phase

Thank you for your attention!