All roads lead to NLSM

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Outline

- NLSM: overview and data
- Amplitudology
- classification: (single) scalar theories
- further avenues

NLSM wasn't built in a day

prehistory: Gell-Mann M, Lévy M ('60), Adler ('65), Weinberg ('66), Susskind, Frye '69, Ellis '70 ...

collaboration with J.Trnka and J.Novotny '13: Soft bootstrap at 10

- motivated by amazing discoveries of amplitudes in gauge theories and gravity (e.g. Parke-Taylor, BCFW) $\rightarrow$ see Jaroslav talk
- we wanted to focus on: Effective field theories
- motivated by theoretical considerations $\rightarrow$ taking something as simple as possible
- very broad subject
- focus on low energy dynamics of theories with SSB
- leading order, tree-level
- strictly massless limit

Leading order Lagrangian

- assume general simple compact Lie group G
- we will build a chiral non-linear sigma model, which will correspond to the spontaneous symmetry breaking ($G_L \simeq G_R \simeq G_V \simeq G$)

$$G_L \times G_R \to G_V$$

- consequence of the symmetry breaking: Goldstone bosons ($\equiv \phi$)

$$U = \exp\left(\sqrt{2}\frac{i}{F}\phi\right)$$

- their dynamics given by a Lagrangian (at leading order)

$$\mathcal{L} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^{-1} \rangle$$

- Using structure constants we can define ordered Feynman rule for the interaction vertices $\rightarrow$ stripped vertices

Stripping and ordering

Up to now general group: we didn't need any property of f^{abc} or t^i.
From now on: we will simplify the problem setting $G = SU(N)$.
Simplification due to the completeness relation:

$$\sum_{a=1}^{N^2-1} \langle X t^a \rangle \langle t^a Y \rangle = \langle XY \rangle - \frac{1}{N} \langle X \rangle \langle Y \rangle$$

- double trace has to cancel out
- two vertices are connected via a propagator (δ^{ab})
- ordering of t^{a_i} in the final single trace is conserved

The tree graphs built form the stripped vertices and propagators are decorated with cyclically ordered external momenta.

$G = U(N)$ – different parametrizations

General form of the parametrization $U(\phi) \to f(x)$

$$f(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad f(-x)f(x) = 1$$

- "exponential": $f_{\text{exp}} = \mathrm{e}^x$
- "minimal": $f_{\text{min}} = x + \sqrt{1 + x^2}$
- "Cayley" $f_{\text{Caley}} = \frac{1+x/2}{1-x/2}$

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$$f(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad f(-x)f(x) = 1$$

- "exponential": $f_{\exp} = e^x$ $\quad \to w_{k,n} = \frac{(-1)^k}{1+\delta_{kn}} \frac{1}{(2n+2)!} \binom{2n+2}{k+1}$
- "minimal": $f_{\min} = x + \sqrt{1 + x^2}$ $\to w_{2k+1,n} = \frac{(-1)^n}{1+\delta_{2k+1,n}} \binom{k-\frac{1}{2}}{k+1} \binom{n-k-\frac{3}{2}}{n-k}$
- "Cayley" $f_{\mathrm{Caley}} = \frac{1+x/2}{1-x/2}$ $\quad \to w_{k,n} = \frac{(-1)^k}{1+\delta_{kn}} \frac{1}{2^{2n}}$

The stripped Feynman rules can be written

$$V_{2n+2}(s_{i,j}) = (-1)^n \left(\frac{2}{F^2}\right)^n \sum_{k=0}^{n} w_{k,n} \sum_{i=1}^{2n+2} s_{i,i+k}$$

where $s_{i,j} \equiv (p_i + p_{i+1} + \ldots + p_j)^2$.

Explicit example: stripped 4pt amplitude

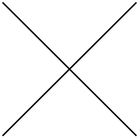
Natural parametrization for diagrammatic calculations: minimal

$$w^{\text{min}}_{2k,n} = 0$$

Thus off-shell and on-shell stripped vertices are equal.

4pt amplitude

$$2F^2\mathcal{M}(1,2,3,4) = -(s_{1,2} + s_{2,3})$$

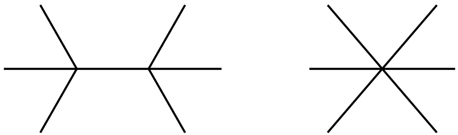


Explicit example: stripped 6pt amplitude

$$
\begin{aligned}
&4F^4\mathcal{M}(1,2,3,4,5,6) = \\
&= \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} + \frac{(s_{1,4}+s_{2,5})(s_{2,3}+s_{3,4})}{s_{2,4}} \\
&\quad + \frac{(s_{1,2}+s_{2,5})(s_{3,4}+s_{4,5})}{s_{3,5}} - (s_{1,2}+s_{1,4}+s_{2,3}+s_{2,5}+s_{3,4}+s_{4,5})
\end{aligned}
$$

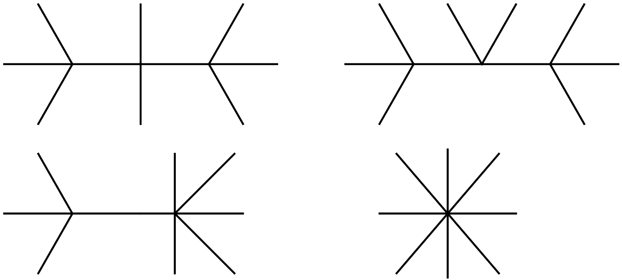
This can be rewritten as

$$
4F^4\mathcal{M}(1,2,3,4,5,6) = \frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} - s_{1,2} + \text{cycl}\,,
$$

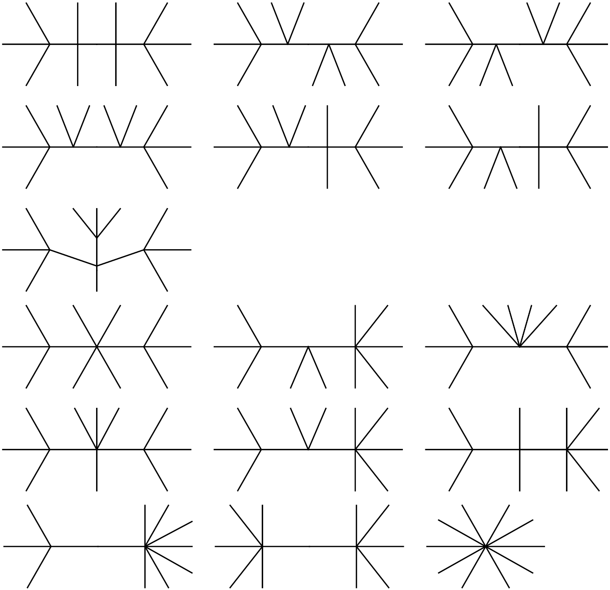


Explicit example: stripped 8pt amplitude

$$8F^6\mathcal{M}(1,2,3,4,5,6,7) =$$
$$= -\frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,7})(s_{5,6}+s_{6,7})}{s_{1,3}s_{5,7}} - \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})(s_{6,7}+s_{7,8})}{s_{1,3}s_{6,8}}$$
$$+ \frac{(s_{1,2}+s_{2,3})(s_{4,5}+s_{4,7}+s_{5,6}+s_{5,8}+s_{6,7}+s_{7,8})}{s_{1,3}} - 2s_{1,2} - \frac{1}{2}s_{1,4} + \text{cycl}$$



Explicit example: stripped 10pt amplitude



Generally: Effective field theories

- very broad subject
- focus on low energy dynamics of theories with SSB
- strictly massless theories
- ground state spontaneously breaks a global symmetry of the underlying theory

$$G \to H$$

- we have Nambu-Goldstone bosons ϕ in the spectrum with

$$\langle 0|J^{\mu}|\phi\rangle \neq 0$$

- $\Rightarrow$ the shift symmetry

$$\phi \to \phi + a$$

- $\Rightarrow$ Adler zero, i.e. vanishing of amplitudes in soft limit

Generally: Effective field theories

Our aim: classification of interesting EFTs

Usual steps:

Symmetry $\to$ Lagrangian $\to$ Amplitudes $\to$ physical quantities
(cross-section, masses, decay constants, ...)

Our method: Amplitudology

works done in collaborations with Christoph Bartsch, Johan Bijnens, Taro Brown, Clifford Cheung, Jiri Novotny, Umut Oktem, Shruti Paranjape, Filip Preucil, Chia-Hsien Shen, Mikhail Shifman, Mattias Sjö, Jaroslav Trnka, Petr Vasko, Congkao Wen...

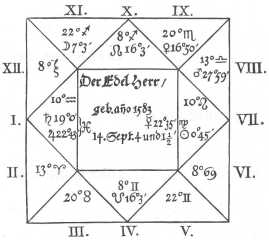
Amplitudology

Not to be confused with astrology...

Amplitudology

Not to be confused with astrology... well, maybe some similarities:
need for precise data [Tycho Brahe] led to

$\rightarrow$ horoscopes [e.g. Kepler for Wellenstein]



Wellenstein's death by K. Piloty

but more importantly to

$\rightarrow$ serious astrophysics [Kepler's laws]

Tycho Brahe's motto

By looking up I see downward

By looking down I see upward

EFT: simplest case

- focus on two derivatives: $\partial_\mu \phi \partial^\mu \phi \phi^n$
- Single field is a trivial case $\rightarrow$ have to consider multi-flavours $\phi_1, \phi_2 \ldots$
- case by case studies: of two, three, ... flavours

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \lambda_{ijkl}\partial_\mu \phi^i \partial^\mu \phi^j \phi^k \phi^l + \lambda_{i_1 \ldots i_6}\partial_\mu \phi^{i_1} \partial^\mu \phi^{i_2} \phi^{i_3} \ldots \phi^{i_6} + \ldots$$

- Very complicated generally
- Assume some simplification using the group structure

$$\phi = \phi^a T^a$$

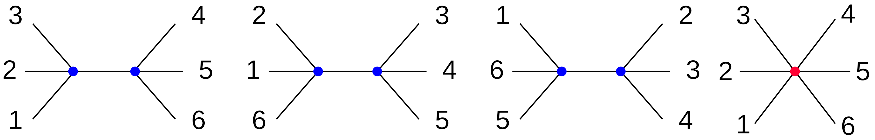
- similar to the 'gluon case': flavour ordering

$$A^{a_1 \ldots a_n} = \sum_{perm} \mathrm{Tr}(T^{a_1} \ldots T^{a_n}) A(p_1, \ldots p_n)$$

First example: NLSM

[KK, Novotny, Trnka '13]

bottom-up analysis, first non-trivial case, the 6pt amplitude:



power-counting:

$$\lambda_4^2\ p^2 \frac{1}{p^2} p^2 + \lambda_6\ p^2$$

in order to combine the pole and contact terms we need to consider some limit. The most natural candidate: we will demand soft limit, i.e.

$$A \to 0, \qquad \text{for} \quad p \to 0$$

$$\Rightarrow \quad \lambda_4^2 \sim \lambda_6 \qquad \text{corresponds to NLSM}$$

How to extend it to all orders (n-pt)? $\to$ new recursion relations

New recursion relations: modification of BCFW

[Cheung, KK, Novotny, Shen, Trnka '15]

The high-energy behaviour forbids a naive Cauchy formula

$$A(z) \neq 0 \qquad \text{for} \quad z \to \infty$$

Can we instead use the soft limit directly?

New recursion relations: modification of BCFW

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Can we instead use the soft limit directly? $\to$ yes!
The standard BCFW not applicable, we propose new shifts:

$$p_i \to p_i(1 - za_i) \qquad \text{on all external legs}$$

This leads to a modified Cauchy formula

$$\oint \frac{dz}{z} \frac{A(z)}{\Pi_i (1 - a_i z)^\sigma} = 0$$

note there are no poles at $z = 1/a_i$ (by construction).

Natural classification: σ and ρ

Generalization of the soft limit:

$$A(tp_1, p_2, \ldots, p_n) = \mathcal{O}(t^{\sigma}), \qquad \text{as} \quad tp_1 \to 0$$

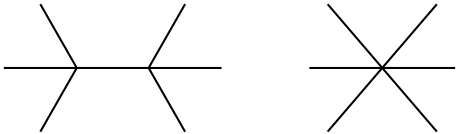
Interaction term

$$\mathcal{L} = \partial^m \phi^n$$

Then another natural parameter is:

$$\rho = \frac{m-2}{n-2} \qquad \text{"averaging number of derivatives"}$$

e.g. $\mathcal{L} = \partial^m \phi^4 + \partial^{\widetilde{m}} \phi^6$



so these two diagrams can mix if the same ρ

Non-trivial cases

$$\text{for: } \mathcal{L} = \partial^m \phi^n : \qquad m < \sigma n$$

or

$$\sigma > \frac{(n-2)\rho + 2}{n}$$

i.e.

ρ	σ at least
0	1
1	2
2	2
3	3

i.e. non-trivial regime for $\rho \leq \sigma$

First case: $\rho = 0$ (i.e. two derivatives)

Schematically for a single scalar case

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \sum_i \lambda_4^i(\partial^2\phi^4) + \sum_i \lambda_6^i(\partial^2\phi^6) + \dots$$

similarly for multi-flavour (ϕ_i: $\phi_1, \phi_2, \dots$).
non-trivial case

$$\sigma = 1$$

Outcome:

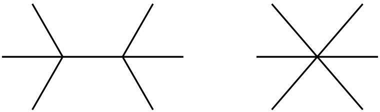
- single scalar: free theory
- multiple scalars (with flavour-ordering): non-linear sigma model

n.b. it represents a generalization of [Susskind, Frye '70], [Ellis, Renner '70]

Second case: $\rho = 1$, $\sigma = 2$ (double soft limit)

1. focus on the lowest combination and fix the form:

$$\mathcal{L}_{int} = c_2(\partial\phi\cdot\partial\phi)^2 + c_3(\partial\phi\cdot\partial\phi)^3 \qquad \text{condition: } c_3 = 4c_2^4$$



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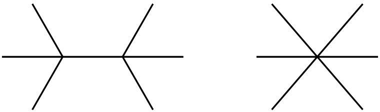

2. find the symmetry

$$\phi \to \phi - b_\rho x^\rho + b_\rho \partial^\rho \phi\, \phi \quad \text{(again up to 6pt so far)}$$

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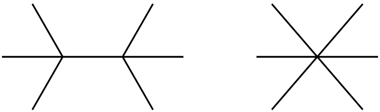

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$$\phi \to \phi - b_\rho x^\rho + b_\rho \partial^\rho \phi \, \phi \quad \text{(again up to 6pt so far)}$$

3. ansatz of the form

$$c_n(\partial\phi \cdot \partial\phi)^n + c_{n+1}(\partial\phi \cdot \partial\phi)^n \partial\phi \cdot \partial\phi$$

4. in order to cancel: $2(n+1)c_{n+1} = (2n-1)c_n$
   i.e. $c_1 = \frac{1}{2} \Rightarrow c_2 = \frac{1}{8}, c_3 = \frac{1}{16}, c_4 = \frac{5}{128}, \ldots$

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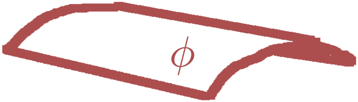
solution:

$$\mathcal{L} = -\sqrt{1 - (\partial\phi \cdot \partial\phi)}$$

This theory known as a scalar part of the Dirac-Born-Infeld [1934] – DBI action

Scalar field can be seen as a fluctuation of a 4-dim brane in five-dim Minkowski space



Third case: $\rho = 2$, $\sigma = 2$ (double soft limit)

Similarly to the previous case, we get a unique solution: the Galileon Lagrangian

$$\mathcal{L} = \sum_{n=1}^{d+1} d_n \phi \mathcal{L}^{\mathrm{der}}_{n-1}$$

$$\mathcal{L}^{\mathrm{der}}_n = \varepsilon^{\mu_1 \ldots \mu_d} \varepsilon^{\nu_1 \ldots \nu_d} \prod_{i=1}^{n} \partial_{\mu_i} \partial_{\nu_i} \phi \prod_{j=n+1}^{d} \eta_{\mu_j \nu_j} = -(d-n)! \det \left\{ \partial^{\nu_i} \partial_{\nu_j} \phi \right\}.$$

It possesses the Galilean shift symmetry

$$\phi \to \phi + a + b_\mu x^\mu$$

and leads to EoM of second-order in field derivatives.

Galileon itself is a remarkable theory: can be connected with a local modification of gravity [Nicolis, Rattazzi, Trincherini '09].

Surprise: $\rho = 2$, $\sigma = 3$ (enhanced soft limit)

- general galileon: three parameters (in 4D)
- only two relevant (due to dualities [de Rham, Keltner, Tolley '14] [KK, Novotny '14])

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- let us demand $\mathcal{O}(p^3)$ behaviour
- we have verified: possible up to very high-pt order
- suggested a new theory: special galileon [Cheung,KK,Novotny,Trnka 1412.4095]

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- suggested a new theory: special galileon [Cheung,KK,Novotny,Trnka 1412.4095]
- symmetry explanation: hidden symmetry [K. Hinterbichler and A. Joyce 1501.07600]

$$\phi \to \phi + s_{\mu\nu} x^\mu x^\nu - 12\lambda_4 s^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

- theory appears also in the context of CHY-type formulation [Cachazo, He, Yuan 1412.3479]

Summary of Classification of EFTs: “soft-bootstrap”

Non-trivial cases

$$\text{for: } \mathcal{L} = \partial^m \phi^n : \qquad m < \sigma n \;\Leftrightarrow\; \sigma > \frac{(n-2)\rho + 2}{n}$$

i.e.

ρ	σ at least
0	1
1	2
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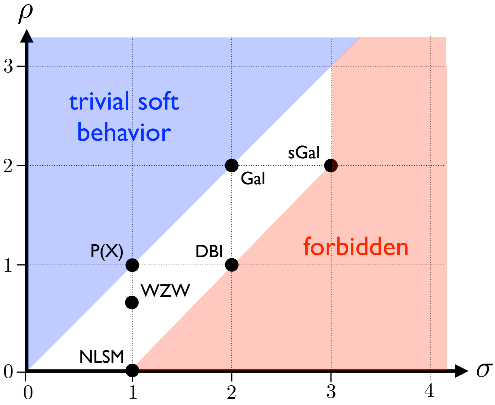
3	3

non-trivial regime for $\rho \leq \sigma$

[C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka '17]

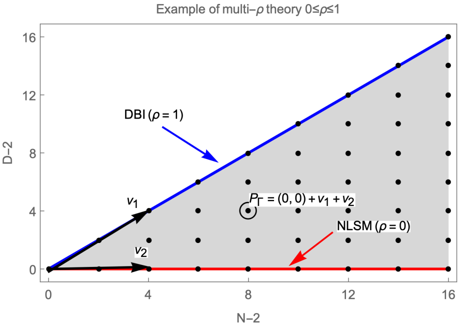


Further directions of the soft amplitudology:

- vector effective field theories from soft limits [1801.01496]
- generalization for Adler zero [1910.04766]
- scalar-vector galileon [2104.10693]
- graded soft theorems [2107.04587]
- higher orders [2109.11574]
- NLSM at one-loop [2206.04694]
  ongoing collaboration with Ch.Bartsch, J.Novotny, J.Trnka on NLSM at all-loop order
- GB on celestial sphere: [2303.14761]
- scalar BCJ bootstrap: [2305.05688]
  ongoing collaboration on exploring the KLT double copy properties

Example of multipower-counting theory

[KK,Novotny,Vasko'21]:



full reconstructibility if: $\rho_{\text{min}} \leq \sigma_{\text{min}} \geq \rho_{\text{max}} \leq \sigma_{\text{max}}$

- Similarly, for DBI-Galileon: interesting possibility of UV completion
- complicated problem: [Adams,Arkani-Hamed,Dubovsky,Nicolis,Rattazzi'06], [Keltner,Tolley'15], ..., [Buoninfante, Tokuda, Yamaguchi'23]

NLSM in double-copy studies

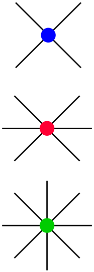
[screenshot from: Z.Bern et al. 1909.01358]:

Double copy	Starting theories	Refs.	Variants and notes
DBI theory	• NLSM • (S)YM theory	[125, 126, 285, 298–301]	• $\mathcal{N} \leq 4$ possible • also obtained as $\alpha' \to 0$ limit of abelian Z-theory
Volkov-Akulov theory	• NLSM • SYM theory (external fermions)	[125, 302–308]	• restriction to external fermions from supersymmetric DBI
Special Galileon theory	• NLSM • NLSM	[125, 285, 301, 306, 309]	• theory is also characterized by its soft limits
DBI + (S)YM theory	• NLSM + ϕ^3 • (S)YM theory	[125, 126, 156, 285, 298–300, 306, 310]	• $\mathcal{N} \leq 4$ possible • also obtained as $\alpha' \to 0$ limit of semi-abelianized Z-theory
DBI + NLSM theory	• NLSM • YM + ϕ^3 theory	[125, 126, 156, 285, 298–300]	

Table 6: List of non-gravitational theories constructed as double copies.

Higher-orders NLSM

40 years of ChPT: up to NNNLO $O(p^8)$
from the amplitude perspective?
yes!: [Dai, Low, Mehen, Mohapatra '20], [KK '21]



	#mesons	#terms
p^2	4	1
p^4	4	2
p^6	4	2
	6	5
p^8	4	3
	6	22
	8	17

Higher-orders NLSM: scalar BCJ bootstrap

[Brown,KK,Oktem,Paranjape, Trnka '23]

BCJ

$$\sum_{i=2}^{n-1}(s_{12}+\ldots+s_{1i})A_n(2,\ldots,i,1,i+1,\ldots,n) = 0\,,$$

We focused on the statement [Gonzalez, Penco, Trodden'19]:

$$\text{BCJ} \;\Rightarrow\; \text{Adler.}$$

For recent studies of the KLT bootstrap see also [Chi, Elvang, Herderschee, Jones, Paranjape '21], [Chen, Elvang, Herderschee '23]

Higher-orders NLSM: scalar BCJ bootstrap

[Brown,kk,Oktem,Paranjape, Trnka '23]

- 4pt

$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	2	3	3	4	4	5	5
BCJ amplitudes	1	0	1	1	1	1	2	1	2

not the final answer!

Higher-orders NLSM: scalar BCJ bootstrap

[Brown,kk,Oktem,Paranjape, Trnka '23]

- 4pt

$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	2	3	3	4	4	5	5
BCJ amplitudes	1	0	1	1	~~1~~ **0**	1	~~2~~ **1**	1	~~2~~ **1**

not the final answer!

- analysis of 6pt (up to $O(p^{18})$ and 8pt (up to $O(p^{10})$): many surprised relations among coefficients of different orders, e.g.

$$\alpha^{(10)} \sim \left(\alpha^{(6)}\right)^2$$

- what are "BCJ Lagrangians"?
  - NLSM
  - Z-theory [Broedel, Schlotterer, Stieberger '13], [Carrasco, Mafra, Schlotterer'16]

Summary

- short overview of ten years of the soft bootstrap
- NLSM represents a role model
- same methods for other theories (DBI, Galileon)
- new theory discovered: special Galileon
- many avenues, e.g.: multi powercounting, double-copy studies, higher-orders
- new surprising connections of NLSM with the $\langle \phi^3 \rangle$ theory up to all orders (!) [Arkani-Hamed et al, in preparation]

thank you!