IDEAL characterizations of spacetimes in cosmology and beyond

(arXiv:1704.05542,1807.09699)

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CEICO Cosmology Meeting 2023
FZU, Prague
11 Oct 2023

Motivation

- **Old observation** (1800s, Riemann, Christoffel, ...):

$$\Delta^{(\Lambda)}_{ijkh}[g] := R_{ijkh}[g] - \alpha(g_{ik}g_{jh} - g_{jk}g_{ih}) = 0 \quad \Longrightarrow \quad g \sim_{\text{loc.isom.}} \text{(A)dS}_\Lambda$$

  with cosmological constant $\alpha \sim \Lambda$ (dim.dep. normalization).
- **Applications:**
  - can **recognize** (A)dS$_\Lambda$ spacetime in any coordinates
  - the linearization $\dot{\Delta}^{(\Lambda)}[h]$ captures **all local linear observables** for metric perturbations h_{ij} on (A)dS, because

$$\dot{\Delta}^{(\Lambda)}[\mathcal{L}_v g] = \mathcal{L}_v \Delta^{(\Lambda)}[g] \quad (= 0 \text{ on (A)dS}_\Lambda)$$

  - writing

$$\Delta^{(\Lambda)}[g] = G^{(\Lambda)}[g] \oplus W^{(\Lambda)}[g],$$

    with $G^{(\Lambda)}[g] = 0$ the Einstein equations, implies that $W^{(\Lambda)}[g] = 0$ on a Cauchy surface Σ generates the **initial data conditions guaranteeing** that the initial data for g on Σ will evolve precisely to (A)dS$_\Lambda$, for arbitrary Σ

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IDEAL Characterization

- **Q:** Given your favorite metric g_0, does there exist a list of **tensors covariantly constructed** from a metric g, $T_1[g], \ldots T_N[g]$, such that
$$\{T_A[g] = 0\}_{A=1}^N \quad \Longrightarrow \quad g \sim_{\text{loc.isom.}} g_0 \,?$$
- When it exists, the list $\{T_A[g] = 0\}_{A=1}^N$ is called an **IDEAL characterization** of the spacetime metric g_0 (**I**ntrinsic, **D**eductive, **E**xplicit, **AL**gorithmic — terminology of Ferrando & Sáez from Valencia).
- **Applications:** $\{T_A[g]\}_{A=1}^N$ is to g_0 like $\Delta^{(\Lambda)}[g]$ is to (A)dS$_\Lambda$
  - recognize g_0
  - recognize initial data for g_0
  - local linear observables for linearized gravity on g_0
- In the presence of matter, we can generalize to $\{T_A[g, \Phi] = 0\}_{A=1}^N$ to characterize metric-matter configurations (g_0, Φ_0).

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Examples

Known IDEAL characterizations is a small, slowly growing list.

- Ferrando & Sáez *et al.*:
  - **Schwarzschild** in 4-dim (1998)
  - **Reissner-Nordström** in 4-dim (2002)
  - **Kerr** in 4-dim (2009)
  - a few more (2010, 2017, . . .)
- (Inflationary) **FLRW** in *n*-dim (2018, Canepa, Dappiaggi & IK)
- **Schwarzschild** in *n*-dim (2019, IK)
- **pp-waves** in 4-dim (2023(?), IK, McNutt & Wylleman)

Rough General Strategy

- Fix a class of reference geometries $(M, g_0(\beta))$, with parameters β.
- Suppose there already exists a characterization of this class by the **existence** of tensor fields σ satisfying equations

$$S_A[g, \sigma] = 0,$$

covariantly constructed from σ, g_{ij}, R_{ijkl} and their covariant derivatives.
- Exploiting the geometry of $(M, g_0(\lambda))$, we try to **find formulas** for $\sigma = \Sigma[g_0]$ covariantly constructed from g_{ij}, R_{ijkl} and their covariant derivatives. If successful, we get an IDEAL characterization of **this class** by

$$T_A[g] := S_A[g, \Sigma[g]] = 0.$$

- If necessary, find **further covariant expressions** for the parameters $\beta = B[g_0]$, adding equations $B[g] - \beta = 0$ to the above list, until we can IDEALly characterize **individual isometry classes.**

FLRW and Inflationary Spacetimes

Let $\dim M = m + 1$.

- (M, g) is (locally) **FLRW** when around every point of M there exist local coordinates $(t, x_1, \ldots, x_m)$, such that
  (a) $g_{ij}(t, x_1, \ldots, x_m) = -(\mathrm{d}t)^2_{ij} + f^2(t) h_{ij}(x_1, \ldots, x_m)$ (**warped product**),
  (b) h_{ij} is of **constant curvature** (**homogeneous** and **isotropic**), e.g.

$$h_{ij} = \frac{1}{(1 - \alpha r^2)} (\mathrm{d}r)^2_{ij} + r^2 \mathrm{d}\Omega^2_{ij}, \quad \text{with} \quad \mathcal{R}[h] = m(m-1)\alpha.$$

- (M, g, ϕ) is (locally) **inflationary** when it is locally **FLRW** and the local coordinates $(t, x_1, \ldots, x_m)$ can be chosen so that the scalar $\phi = \phi(t)$, while also satisfying the **Einstein-Klein-Gordon** equations

$$R_{ij} - \frac{1}{2}\mathcal{R} g_{ij} = \kappa \left(\nabla_i \phi \nabla_j \phi - \frac{1}{2} g_{ij} [(\nabla\phi)^2 + V(\phi)] \right)$$

  with some potential $V(\phi)$ and $\kappa \sim$ Newton's constant.

Tensors for Warped $(m+1)$-dim Products

A Generalized Robertson Walker (**GRW**) $(m+1)$-dim geometry is like FLRW, but with no conditions on spatial slices.

Theorem (Sánchez, 1998)

(M,g) is locally GRW iff $\exists U$ — unit timelike vector field satisfying

$$\mathfrak{P}_{jk} := U_{[j}\nabla_{k]}\frac{\nabla^i U_i}{m} = 0, \quad \mathfrak{D}_{ij} := \nabla_i U_j - \frac{\nabla_k U^k}{m}(g_{ij} + U_i U_j) = 0.$$

In coordinates, $U^i = (\partial_t)^i$, for instance $U = \nabla\phi/\sqrt{-(\nabla\phi)^2}$.

GRW **pre-history**: in 4-dim cosmology, Sánchez's conditions were know much earlier (Ehlers, 1961; Easley, 1991). They were stated as follows: *U is unit, geodesic, shear-free, twist-free and has spatially-constant expansion*.

(Ellis & Bruni, 1989): used linearizations $\dot{\mathfrak{P}}[h]$ and $\dot{\mathfrak{D}}[h]$ to build linearized observables in cosmology

Tensors for Constant Spatial Curvature

- Convenient to define the Kulkarni-Nomizu product:

$$(A \odot B)_{ijkh} = A_{ik} B_{jh} - A_{jk} B_{ih} - A_{ih} B_{jk} + A_{jh} B_{ik}.$$

- Given Sánchez's U^i, define $\xi := \frac{\nabla^i U_i}{m}$, $\eta := -U^i \nabla_i \xi$.
  Eventually, U is one of normalized $\nabla\mathcal{R}$, $\nabla(\mathcal{B} := R_{ij} R^{ij})$ or $\nabla\phi$.
- GRW Spatial Zero Curvature Deviation (**ZCD**) tensor:

$$\mathfrak{Z}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{\xi^2}{2} g - \eta UU\right]\right)_{ijkh}, \quad \zeta := \frac{\mathfrak{Z}_i{}^i{}_k{}^k}{m(m-1)}.$$

- GRW Spatial Constant Curvature Deviation (**CCD**) tensor:

$$\mathfrak{C}_{ijkh} := R_{ijkh} - \left(g \odot \left[\frac{(\xi^2 + \zeta)}{2} g - (\eta - \zeta) UU\right]\right)_{ijkh}.$$

- $\mathfrak{Z}_{ijkh} = 0 \implies$ **flat FLRW.**
  $\mathfrak{C}_{ijkh} = 0$, $U_{[i} \nabla_{j]} \zeta = 0 \implies$ **generic FLRW** (any curvature).

Inflationary FLRW Scale Factor

- Scale factor derivatives: $\xi = \frac{f'}{f}, \quad \boldsymbol{\eta} = \frac{f''}{f} - \frac{f'^2}{f^2}, \quad \zeta = \frac{\alpha}{f^2}.$
- **Einstein-Klein-Gordon** equations reduce to

$$\xi^2 + \zeta = \kappa \frac{\phi'^2 + V(\phi)}{m(m-1)}, \quad \boldsymbol{\eta} - \zeta = -\kappa \frac{\phi'^2}{(m-1)}.$$

- Flat inflationary with $\zeta = 0$, $\phi' \neq 0$: can find $\Xi(u)$ such that

("Hamilton-Jacobi" eq. *á la* Salopek & Bond, 1990)

$$(\partial_u \Xi(u))^2 - \kappa \frac{m\Xi^2(u)}{(m-1)} + \kappa^2 \frac{V(u)}{(m-1)^2} = 0,$$

$$\phi' = -\frac{(m-1)}{\kappa} \partial_\phi \Xi(\phi), \quad \xi = \Xi(\phi).$$

- Generic inflationary with $\phi' \neq 0$: can find $\Xi(u)$, $\Pi(u)$ such that

(new?)

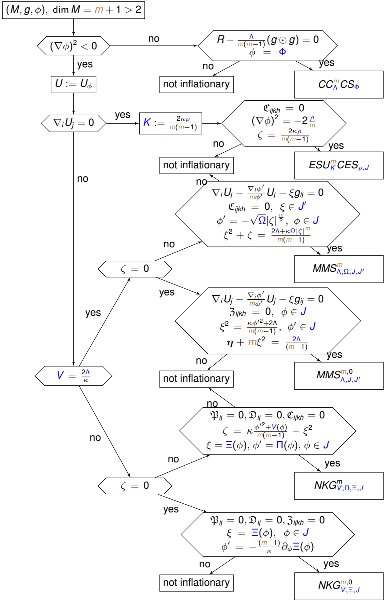
$$\Pi \left(\partial_u \Xi + \kappa \frac{\Pi}{(m-1)} \right) - \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) = 0,$$

$$\partial_u \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) + 2\frac{\Xi}{\Pi} \left(\kappa \frac{\Pi^2 + V}{m(m-1)} - \Xi^2 \right) = 0,$$

$$\phi' = \Pi(\phi), \quad \xi = \Xi(\phi).$$

- ODEs in (f, ϕ) **fix scale factor and inflaton** up to $(f(t), \phi(t)) \mapsto (Af(t + t_0), \phi(t + t_0))$, exhausting isometric $(f(t), \phi(t))$ pairs.

Flowchart: Inflationary Characterization



Discussion

- An IDEAL characterization of the (local) isometry class of a physically interesting spacetime is a **natural geometric problem** with applications to linear observables and classification of exact solutions in gravity.
- Some **cosmological**, **black hole** and now also **pp-wave** spacetimes have IDEAL characterizations.
- The Cartan-Karlhede **moving frame** method is an older and more developed method of invariant characterization of geometries. Does an IDEAL characterization **exist** whenever a Cartan-Karlhede characterization exists? (No-ish.)
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Thank you for your attention!