

Exploring Quasinormal Modes of Black Holes: Efficient Computation Techniques Using Publicly Shared Code

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October 12, 2023

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1. Quasinormal modes: Introduction.

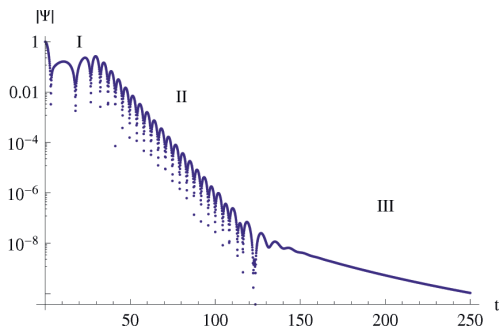


Figure: An example of the time-domain profile for the $\ell = 2$ axial gravitational perturbations of the Schwarzschild black hole.

- (I) - initial outbursts;
- (II) - quasinormal ringing;
- (III) - power-law tails.

2. Perturbations, wave equations, boundary conditions.

1 Test fields:

- 1 general covariant equation for a given test field in curved spacetime;
- 2 separation of variables, leading to a wave-like equation

2 Gravitational field:

- 1 The perturbed metric has the form

$$g_{\mu\nu} = g_{\mu\nu 0} + \delta g_{\mu\nu}, \quad (1)$$

where $g_{\mu\nu 0}$ is the background (unperturbed) metric; $\delta g_{\mu\nu}$ is perturbation;

- 2 linearization of the dynamical equations for the gravitational and matter fields leading, after separation of variables to wave-like equation for a gauge-invariant combination from $\delta g_{\mu\nu}$.

In the majority of cases, one can reduce perturbation equations to the wave-like form:

$$\frac{d^2\Psi}{dx^2} = U(x, \omega)\Psi, \quad (2)$$

where in the simplest (but popular) case $U(x, \omega) = V(x) - \omega^2$, $V(x)$ is the effective potential, x is the tortoise coordinate, $\omega = \text{Re}(\omega) + i\text{Im}(\omega)$, $\text{Im}(\omega) \sim$ damping rate.
The boundary conditions:

- 1 $x \rightarrow -\infty \Rightarrow$
a purely ingoing wave at the event horizon,
- 2 $x \rightarrow +\infty \Rightarrow$
a purely outgoing wave at infinity.

3. One-slide-review of SOME methods for calculating QNMs.

- 1 **Frobenius method.** = Leaver method = Continued fraction method. It uses the expansion of the solution for the wave function into the Frobenius series. Based on the convergent procedure and allows precise calculation of QNMs. Requires analysis of singular points of differential equations and an initial guess for the value of the frequency. [First: E. Leaver, Proc.Roy.Soc.Lond.A 402 (1985) 285-298]
- 2 **Integration in time-domain.** Construction of the time-domain profile at a fixed spacial coordinate like on the fig. 1 above. Include contributions of all overtones at a given ℓ . Extraction of frequencies from the profile is possible for a few lowest modes by the Prony method. It allows one to detect the onset of instability. [First: C. Gundlach, R. H. Price, J. Pullin, Phys.Rev.D 49 (1994) 883-889]
- 3 **WKB method** is a semi-analytic method, which allows one to obtain automatically QNM with sufficiently high accuracy for a broad range of black hole configurations. It has a [publicly shared code](#) in [R. K, A. Zhidenko, and A. F. Zinhailo, Class. Quant. Grav., 36:155002, 2019] and a number of limitations. [First: B. Schutz, C. Will, Astrophys. J. Lett. 291 (1985) L33-L36]
- 4 **Bernstein spectral method.** The wave function is represented as a sum over the Bernstein polynomials. The method is automatically applied to various spacetimes and has a [publicly shared code](#) in [R. A. K., A. Zhidenko, Phys.Rev.D 107 (2023) 4, 044009]. Efficient for finding several lowest modes and detecting instability. [First: S. Fortuna and I. Vega, [arXiv:2003.06232 [gr-qc]]]

4. WKB method: a bit of chronology.

- 1 The first WKB order. B. Mashhoon, in 3rd Marcel Grossmann Meeting, p. 598, 1982; B. Schutz, C. Will, *Astrophys. J. Lett.*, 291:L33–L36, 1985.
- 2 The 3d-order of WKB formula S. Iyer and C. Will *Phys. Rev. D*, 35:3621, 1987.
- 3 The 6th-order of WKB formula R. K., *Phys. Rev. D*, 68:024018, 2003.
- 4 13th order and Padé approximants J. Matyjasek and M. Opala, *Phys. Rev. D*, 96:024011, 2017.

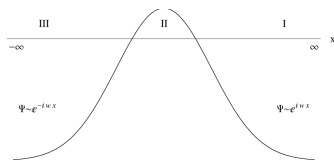


Figure: The three regions separated by the two turning points.

- (I, III) - WKB series;
(II) - Taylor series.

4. WKB method.

The result of the matching of the WKB and Taylor series is the general form:

$$\begin{aligned} \omega^2 = & V_0 + A_2(K^2) + A_4(K^2) + A_6(K^2) + \dots \\ & -iK\sqrt{-2V_2(\omega)} (1 + A_3(K^2) + A_5(K^2) + A_7(K^2) \dots), \end{aligned} \quad (3)$$

where $A_k(K^2)$ is the correction of order k to the eikonal formula, V_2 is the second derivative of the effective potential in its maximum relatively the tortoise coordinate. It can be found for each order separately. For the first order we have $A_2 = A_3 = A_4 = \dots = 0$ and:

$$\omega^2 = V_0 - iK\sqrt{-2V_2}, \quad (4)$$

For boundary condition of quasinormal problem:

$$K = \begin{cases} +n + \frac{1}{2}, & \omega_{Re} > 0; \\ -n - \frac{1}{2}, & \omega_{Re} < 0; \end{cases} \quad (5)$$

$n = 0, 1, 2, 3, \dots$

The WKB formula is exact in the eikonal limit $\ell \rightarrow \infty$ and usually provides very good accuracy for $\ell \gg n$.

4. WKB method: Restrictions.

The WKB formula impose strict limits on the range of its applicability. In particular, the WKB formula cannot be applied when studying:

- 1 **Superradiance.** Owing to extracting rotational energy from a black hole, the incident wave can be reflected with larger amplitude than it had in the beginning. It also takes place for a charged scalar field in the non-rotating electrically charged black-hole background.
- 2 **Stability.** Unstable modes $\omega_{lm} > 0$ correspond to the bound states, and the analytic continuation in this case should be done in a different way.
- 3 **Infinitely long-lived modes, called *quasiresonances*.** For large values of the field mass, the effective potential does not have a local maximum, so that the WKB expansion cannot be performed.
- 4 **Higher overtones of the quasinormal spectrum.** Usually the WKB accuracy is reasonable for $\ell > n$ and marginal already for $\ell = n$, ℓ is a multipole number.
- 5 **Asymptotically nonconstant potential** (for example, for black holes with the anti-de Sitter asymptotic).

5. WKB with Padé approximants.

To improve accuracy one can consider an extension of the WKB-method by using the Padé approximants (Matyjasek and M. Opala, 2017):

$$\begin{aligned} P_{\tilde{n}+\tilde{m}}(\epsilon) &= P_k(\epsilon) = V_0 + A_2(K^2)\epsilon^2 + A_4(K^2)\epsilon^4 + A_6(K^2)\epsilon^6 + \dots \\ &- iK\sqrt{-2V_2}(\epsilon + A_3(K^2)\epsilon^3 + A_5(K^2)\epsilon^5 \dots), \end{aligned} \quad (6)$$

where k is the polynomial order coincides with the WKB order. Formal parameter ϵ is introduced in order to track WKB orders. With Padé approximants polynomial $P_{\tilde{n}+\tilde{m}}(\epsilon)$ is a rational function:

$$P_{\tilde{n}/\tilde{m}}(\epsilon) = \frac{Q_0 + Q_1\epsilon + \dots + Q_{\tilde{n}}\epsilon^{\tilde{n}}}{R_0 + R_1\epsilon + \dots + R_{\tilde{m}}\epsilon^{\tilde{m}}}, \quad (7)$$

The coefficients $Q_0, Q_1, \dots, Q_{\tilde{n}}$ and $R_0, R_1, \dots, R_{\tilde{m}}$ can be obtained in the same way as A_2, A_3, \dots, A_k by matching the expansion of the solution near the potential peak through the turning points. For first Padé approximation we have,

$$\omega^2 = P_{0/1}(1) = \frac{V_0^2}{V_0 + iK\sqrt{-2V_2}}. \quad (8)$$

Usually, when $\tilde{n} \approx \tilde{m}$, the approximation is much better comparing to the usual WKB formula without Padé approximants.

6. Examples: QNMs.

$method_{order}$	ω_k	$Error_{Re}$	$Error_{Im}$
$\ell = 0, n = 0: \omega = 0.220910 - 0.209791i$			
WKB_1	$0.189785 - 0.098239i$	14.090%	53.173%
WKB_3	$0.209294 - 0.230394i$	5.258%	9.821%
WKB_6	$0.220934 - 0.201633i$	0.011%	3.889%
WKB_7	$0.225845 - 0.207002i$	2.234%	1.329%
WKB_9	$0.256553 - 0.228135i$	16.135%	8.744%
Pad_1	$0.299357 - 0.154959i$	35.511%	26.137%
Pad_3	$0.221995 - 0.200495i$	0.491%	4.431%
Pad_6	$0.222620 - 0.209111i$	0.774%	0.324%
Pad_7	$0.223831 - 0.208727i$	1.322%	0.507%
Pad_9	$0.220627 - 0.209924i$	0.128%	0.0634%

Table: Quasinormal modes of a massless scalar field for $\ell = 0$ calculated with the WKB formula and Padé approximation of different orders: The error $E_{Re/Im} = \left| \frac{\omega_k - \omega}{\omega} \right| \times 100\%$. The frequencies are given in units $r_h = 1$ and for conversion into kHz one should multiply by $\pi(5142\text{Hz}) \cdot (M_\odot/M)$. For example, a black hole of mass $M = 10 \cdot M_\odot$ has a frequency of 1.2 kHz and damping time of 0.55 ms.

6. Examples: QNMs.

$method_{order}$	ω_k	$Error_{Re}$	$Error_{Im}$
$\ell = 1, n = 0: \omega = 0.585872 - 0.195320i$			
WKB_1	$0.655100 - 0.184647i$	11.8163%	5.4642%
WKB_3	$0.582228 - 0.196003i$	0.6220%	0.3497%
WKB_6	$0.585819 - 0.195523i$	0.0090%	0.1039%
WKB_7	$0.585746 - 0.195305i$	0.0215%	0.0077%
WKB_9	$0.585867 - 0.195284i$	0.0009%	0.0271%
Pad_1	$0.658868 - 0.192512i$	12.459%	1.4374%
Pad_3	$0.585345 - 0.194519i$	0.0899%	0.4101%
Pad_6	$0.585864 - 0.195320i$	0.0014%	0.0002%
Pad_7	$0.585867 - 0.195321i$	0.0008%	0.0004%
Pad_9	$0.585868 - 0.195316i$	0.0007%	0.0022%

Table: Quasinormal modes of the massless scalar field for $D = 4$, $\ell = 1$ calculated with the WKB formula and Padé approximation of different orders: The error

$$E_{Re/Im} = \left| \frac{\omega_k - \omega}{\omega} \right| \times 100\%.$$

6. Scattering problem.

The intensity of Hawking radiation is always partially suppressed by the effective potential surrounding the black holes, because the part of the total flow of particles emitted by the black hole is reflected back to the event horizon. In order to estimate the number of particles reflected by the effective potential we need first to find the gray-body factors, that is, to solve the classical scattering problem. Wave equation:

$$\frac{d^2\Psi}{dx^2} = (V(x) - \omega^2)\Psi, \quad (9)$$

with boundary conditions

$$\begin{aligned} \Psi &= e^{-i\omega x} + R e^{i\omega x}, & x \rightarrow +\infty, \\ \Psi &= T e^{-i\omega x}, & x \rightarrow -\infty, \end{aligned} \quad (10)$$

where R and T are the reflection and transmission coefficients. In particular, when the effective potential is real, K is a purely imaginary constant related with the reflection and transmission coefficients in the following way,

$$|R|^2 = \frac{1}{1 + e^{-2\pi i K}}, \quad 0 < |R|^2 < 1. \quad (11)$$

$$|T|^2 = \frac{1}{1 + e^{2\pi i K}} = 1 - |R|^2. \quad (12)$$

6. Scattering problem.

Energy emission rate for Hawking radiation has the form:

$$\frac{dE}{dt} = \sum_{\ell} N_{\ell} |\mathcal{A}_{\ell}|^2 \frac{\omega}{\exp(\omega/T_H) \pm 1} \frac{d\omega}{2\pi}, \quad (13)$$

where \mathcal{A}_{ℓ} are the grey-body factors, and N_{ℓ} are the multiplicities, which only depend on the space-time dimension and ℓ ,

$$T_H = \frac{1}{4\pi} \sqrt{-\frac{g'_{tt}}{g'_{rr}}} \Big|_{r=r_+}$$

is the Hawking temperature for spherically symmetric black hole.

6. Scattering problem.

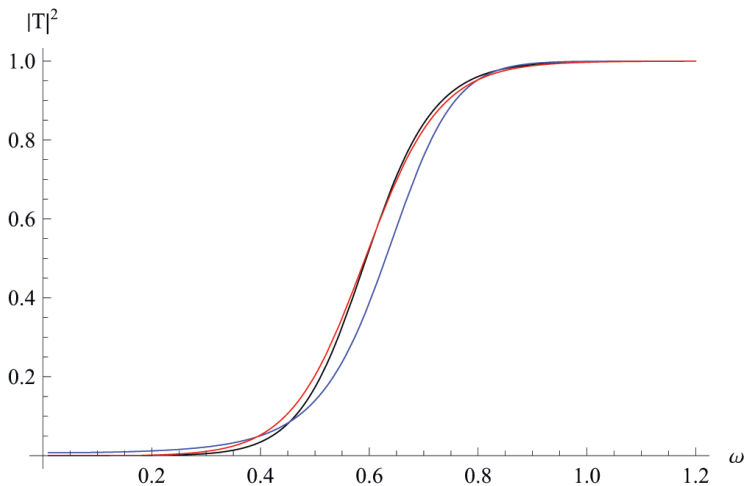


Figure: Transmission coefficients for $\ell = 1$ (black), eikonal approximation (blue), second-order WKB formula (red).

6. Scattering problem.

ℓ/k	dE/dt by Page	dE/dt by WKB	Error
Neutrinos			
$k = 1$	0.000630	0.000639	1.42857%
$k = 2$	0.000024	0.000024	0%
$k = 3$	4×10^{-7}	4.7×10^{-7}	17.5%
The sum	0.00065	0.00066	1.53846%
Photons			
$\ell = 1$	$1.32 \cdot 10^{-4}$	$1.34 \cdot 10^{-4}$	1.51515%
$\ell = 2$	$2.80 \cdot 10^{-6}$	$2.67 \cdot 10^{-6}$	4.64286%
$\ell = 3$	$4 \cdot 10^{-8}$	$4.03 \cdot 10^{-8}$	1.75%
The sum	$1.35 \cdot 10^{-4}$	$1.37 \cdot 10^{-4}$	1.45985%
Gravitons			
$\ell = 2$	$0.152 \cdot 10^{-4}$	$0.152 \cdot 10^{-4}$	0%
$\ell = 3$	$1.6 \cdot 10^{-7}$	$1.43 \cdot 10^{-7}$	10.625%
The sum	$0.154 \cdot 10^{-4}$	$0.154 \cdot 10^{-4}$	0.110677%

Table: Energy emission for the Schwarzschild background ($M = 1/2$) after integrating over all the quantum numbers and ω in Page's work versus that obtained here with the help of the 6th order WKB formula.

7. Bernstein polynomial spectral method.

For asymptotically flat black holes we introduce the compact coordinate u ,

$$u \equiv \frac{1}{r},$$

and represent $y(u)$ as a sum,

$$y(u) = \sum_{k=0}^N C_k B_k^N(u), \quad (14)$$

where

$$B_k^N(u) \equiv \frac{N!}{k!(N-k)!} u^k (1-u)^{N-k}$$

are the Bernstein polynomials. Substituting the wave function into the master wave equation and using a Chebyshev collocation grid of $N + 1$ points, we obtain a set of linear equations with respect to C_k , which has nontrivial solutions iff the corresponding coefficient matrix is singular. The problem is reduced to the eigenvalue problem of a matrix pencil with respect to ω , which can be solved numerically. Once the eigenvalue problem is solved, one can calculate the corresponding coefficients C_k and explicitly determine the polynomial (14), which approximates the solution to the wave equation.

We have considered two automatic methods for which codes are publicly shared, the WKB method with Pade approximants and the Bernstein polynomial method.

- 1 If the effective potential has single maximum and monotonically decay at the boundaries, usually you can find modes $\ell \geq n$ with sufficient accuracy by the 6th or 7th order WKB method with Pade approximants $\tilde{m} \approx \tilde{n}$.

To download the code: <https://docs.google.com/document/d/1Qot9n954AhM-euRy1cBgPx77devbVjoAiyLwLUqHVmA/preview>

- 2 The grey-body factors can be found sufficiently accurately by the usual 6th WKB order formula.
- 3 The first several quasinormal modes with whatever relation between n and ℓ can be found by the Bernstein polynomial method. It is especially effective for detecting purely imaginary modes, such as algebraically special mode, or modes driving the instability.

To download the code: <https://arxiv.org/src/2211.02997/anc>

Thank you for your attention!