The GNS-construction and undetectable non-equilibrium states

Based on ongoing work with Souvik Banerjee

2311.xxxxx

Reasonable measurements in complex systems

The tools of this talk: algebras of observables

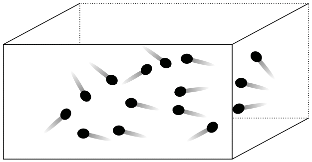
What can we realistically measure in a complex quantum system?

Simple observables:

- One particle positions/momenta
- Positions/momenta of several particles

Complicated observables:

- The positions of <u>all</u> particles simultaneously
- The Hamiltonian(!)



Example: gas in a box $|\Psi\rangle = |\psi_1\rangle ... |\psi_N\rangle$

Undetectable excitations

the goal of this talk:

From the perspective of simple observables:

$$|E\rangle \approx |\tilde{E}\rangle = e^{-\frac{\beta}{2}\hat{H}} U e^{\frac{\beta}{2}\hat{H}} |E\rangle \tag{1}$$

Where:

- $|H\rangle$ is a typical state of a canonical ensemble with inverse temperature β
- U is unitary and $[U, \hat{H}] \neq 0$

Under the assumption of approximate thermal physics[1]:

$$\begin{aligned}
&\langle \tilde{E}|O...O|\tilde{E}\rangle \\
&= \mathrm{Tr}\left(e^{-\beta\hat{H}} e^{\frac{\beta}{2}\hat{H}} U^\dagger e^{-\frac{\beta}{2}\hat{H}} O...O e^{-\frac{\beta}{2}\hat{H}} U e^{\frac{\beta}{2}\hat{H}}\right) + \mathcal{O}(e^{-S/2}) \\
&= \mathrm{Tr}\left(e^{-\beta\hat{H}} O...O\right) + \mathcal{O}(e^{-S/2})
\end{aligned} \tag{2}$$

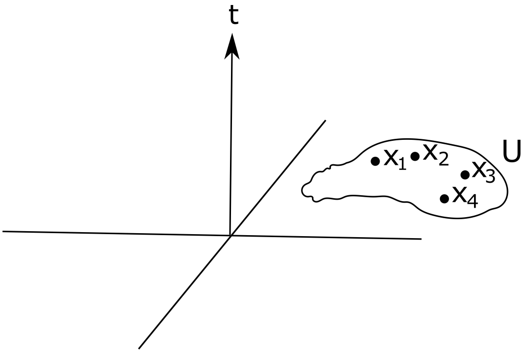
We will derive this from first principles in conformal field theories.

[1] As originally derived in 1708.06328 by K. Papadodimas

Simple, but powerful!

Theorem

Reeh-Schlieder theorem: *The set of vectors* $\{O_1(x_1)...O_n(x_n)|0\rangle\}$ *generated by local fields* $O_i(x_i)$ *where* x_i *are restricted to an open set* U *is dense in the Hilbert space* $\mathcal{H}$*.*



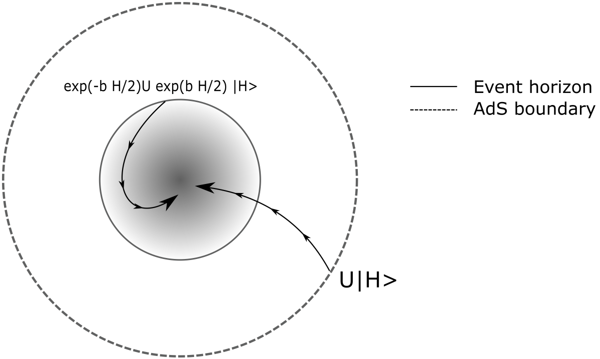
Simple observables have a lot of resolution power!

Application

Application: Holographic models

In holographic CFTs with bulk gravitational duals:

$|H\rangle \longleftrightarrow$ black hole microstates!



GNS-construction

Starting point: a $*-$algebra of observables

$$\text{Span}\left(O_1, ..., O_n, O_1 O_2, ..., O_4^8 O_3 O_8^2, ..., O_1^\dagger, ..., O_n^\dagger\right)$$

Closed under a particular norm $\rightarrow$ von Neumann algebra

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Construct an auxiliary GNS-Hilbert space[2]:

$$O_k \rightarrow |O_k\rangle\rangle$$

[2] GNS: Gelfand, Naimark (1943), Segal (1947)

GNS-linear form

Tool for the job: GNS-linear form $\rightarrow$ $\omega(O) = \#$

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properties

- semi-positivity: $\omega(O^\dagger O) \geq 0$
- (to avoid technicalities) non-degeneracy: $\forall B, \quad \omega(A^\dagger B) = 0 \implies A = 0$

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Construct a Hilbert space ⟶ define the scalar products

$$\langle\langle O_i | O_j \rangle\rangle \equiv \omega(O_i^\dagger O_j)$$

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Construct a Hilbert space $\longrightarrow$ define the scalar products

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Example: Density matrices ρ

$$\omega(O) = \mathrm{Tr}\,(\rho O)$$

Algebra representations

Two inequivalent representations of the operator algebra on the GNS-Hilbert space

$$\pi(O_1)|O_2\rangle\rangle = |O_1 O_2\rangle\rangle$$

$$\pi_r(O_1)|O_2\rangle\rangle = |O_2 O_1^\dagger\rangle\rangle$$

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These representations commute

$$\pi(O_1)\pi_r(O_2)|O_3\rangle\rangle = \pi(O_1)|O_3 O_2^\dagger\rangle\rangle = |O_1 O_3 O_2^\dagger\rangle\rangle$$

$$= \pi_r(O_2)|O_1 O_3\rangle\rangle = \pi_r(O_2)\pi(O_1)|O_3\rangle\rangle$$

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$$|K_0\rangle\rangle \rightarrow \omega(O) = \langle\langle K_0|\pi(O)|K_0\rangle\rangle$$

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The original motivation: thermal expectation values become vacuum expectation values

$$\mathrm{Tr}\left(e^{-\beta H} O_1 ... O_n\right) = \langle\langle \pi(O_1) ... \pi(O_n)\rangle\rangle$$

Part II

Part II: Tomita-Takesaki theory and undetectable excitations

Undetectable excitations

Reminder: The goal: prove that

$$|E\rangle \approx |\tilde{E}\rangle = e^{-\frac{\beta}{2}\hat{H}} U e^{\frac{\beta}{2}\hat{H}} |E\rangle$$

Context:

- In 2d conformal field theory (spacetime symmetry: 2 copies of the infinite dimensional Virasoro algebra)

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{0,n+m}$$

- Large number of local degrees of freedom (technical terms: large central charge c)
- High-energy: $\langle E|\hat{H}|E\rangle = \mathcal{O}(c)$
- 'light' observables: $\langle 0|O_1 ... O_n|0\rangle \ll c$

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Take as GNS-linear form the Euclidean CFT correlator

$$\omega(O) = \langle E|O|E\rangle$$

Modular Hamiltonian

$|E\rangle$ is an energy eigenstate $\longrightarrow$ time-translation invariant

$\Longrightarrow$ one-dimensional automorphism symmetry group parametrized by s

$$\pi(O) \to p_s\pi(O) = e^{i\mathcal{K}s}\pi(O)e^{-i\mathcal{K}s}$$

$$\langle\langle e^{i\mathcal{K}s}\pi(O)e^{-i\mathcal{K}s}\rangle\rangle = \langle\langle \pi(O)\rangle\rangle$$

Modular Hamiltonian: $\mathcal{K}$, generator of the automorphism group

- $\mathcal{K}$ is Hermitian
- $\mathcal{K}$ annihilates the GNS-vacuum: $\mathcal{K}|K_0\rangle\rangle = 0$
- In terms of the system Hamiltonian: $\mathcal{K} = \pi(\hat{H}) - \pi_r(\hat{H})$

KMS-condition

Special property: Through standard CFT techniques $\rightarrow \mathcal{K}$ satisfies the KMS-condition[3]

$$\langle\langle\pi(O_1)\, e^{\beta\mathcal{K}}\pi(O_2)e^{-\beta\mathcal{K}}\rangle\rangle = \langle\langle\pi(O_2)\pi(O_1)\rangle\rangle$$

where

$$\beta = \frac{1}{2\pi}\sqrt{1 - 24\frac{E}{c}}$$

KMS-condition is a combined:

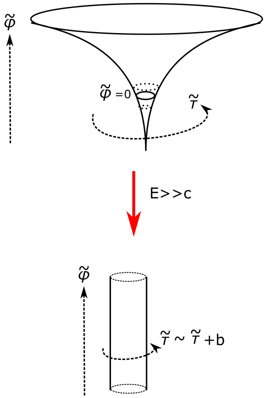
- periodicity condition
- order-reversing property

[3] Kubo (1957), Martin, Schwinger (1959)

CFT sketch

Let's sketch the CFT computation:

- State-operator map $\rightarrow \langle E|O_1 ... O_n|E\rangle = \langle O_H(0) O_H(\infty) O_1 ... O_n\rangle$
- Heavy-Light decoupling $\langle O_H(z_1) O_H(z_2) O_1(x_1) ... O_n(x_n)\rangle \approx f(x_i, h_i|z_i, H)\langle O_H(z_1) O_H(z_2)\rangle$
- Light sector $\rightarrow$ 'couples' to the heavy stress tensor expectation value $T(z) = \langle T(z) O_H(z_1) O_H(z_2)\rangle$
- $T(z)$ transforms inhomogenously $T' = w'(z)^{-2}\left(T - \frac{c}{12} S[w,z]\right)$ where $S[w,z] = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$



Operator ordering

Operator switching

Lorentzian expectation in Heisenberg frame:

$$\langle E|O_n e^{-iH(t_n - t_{n-1})} O_{n-1} \ldots e^{-iH(t_2 - t_1)} O_1 |E\rangle$$

Edge-of-the-Wedge theorem

- $\rightarrow$ complexify time-differences $\tau_k = t_k - t_{k-1}$
- analytic continuation $\rightarrow$
- Expectation value holomorphic in τ_k on lower half-plane!.

$\rightarrow$ **Recipe for operator ordering:**

- set real part of times equal to zero
- set imaginary parts of times so differences are in the lower half-plane
- evaluate resulting Euclidean correlator
- set imaginary parts to zero, set real parts back to their original value

Consequence: Ordering is determined by universal cover of Euclidean time!

(Also) moving to Japan!

Given a Tomita-Takesaki operator S such that

$$|SOK_0\rangle\rangle = |O^\dagger K_0\rangle\rangle$$

S has a unique polar decomposition

$$S = J\Delta^{1/2}$$

where

- J is anti-unitary[4]
- $J^2 = \mathbb{I}$
- Δ is Hermitian
- $|JOk_0\rangle\rangle = |K_0 O^\dagger\rangle\rangle$

Theorem

Tomita-Takesaki theorem: *There is a 1-parameter automorphism group acting on $\pi(O)$ given by $\Delta^{is}\pi(O)\Delta^{-is}$ which satisfies the KMS-condition.*

[4] $\langle\langle O_1 J | J O_2\rangle\rangle = \langle\langle O_2 | O_1\rangle\rangle$

Automorphism uniqueness theorem

Theorem

Automorphism uniqueness theorem: *If an automorphism group $p_s\pi(O) = e^{i\mathcal{K}s}\pi(O)e^{-i\mathcal{K}s}$ satisfies the KMS-condition then $e^{\beta\mathcal{K}}$ has to coincide with Δ.*

Note that $\Delta^{1/2} = JS$ hence

$$|\Delta^{1/2} O K_0\rangle\rangle = |K_0 O\rangle\rangle$$

Since $e^{\frac{\beta}{2}\mathcal{K}} = \Delta^{1/2}$ we find at the operator level

$$e^{-\frac{\beta}{2}\mathcal{K}}\pi(O)e^{\frac{\beta}{2}\mathcal{K}}|K_0\rangle\rangle = \pi_r(O^\dagger)|K_0\rangle\rangle$$

Putting it all together

Putting it all together:

remembering that $\mathcal{K} = \pi(\hat{H}) - \pi_r(\hat{H})$ we find that

$$\langle E| e^{\frac{\beta}{2}\hat{H}} U^\dagger e^{-\frac{\beta}{2}\hat{H}} O_1 ... O_n e^{-\frac{\beta}{2}\hat{H}} U e^{\frac{\beta}{2}\hat{H}} |E\rangle$$

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$$= \langle\langle \pi_r(U) \pi(O_1) ... \pi(O_n) \pi_r(U^\dagger) \rangle\rangle$$

$$= \langle\langle \pi(O_1) ... \pi(O_n) \rangle\rangle = \langle E| O_1 ... O_n |E\rangle$$

conclusion

- One can derive a lot of physics from considering algebras of observables
- Under specific conditions we identified the KMS-condition in CFT correlators
- Without assuming typicality of a thermal ensemble we derived the indistinguishability of $|E\rangle$ and $|\tilde{E}\rangle$

Thank you for you attention